Factorization homology of orbifolds

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Abstract. We construct a factorization homology of orbifolds, modelled as separated étale stacks on the category of smooth manifolds. This extends the definition of factorization homology of smooth manifolds introduced as topological chiral homology in [Lur]. We prove that factorization homology of orbifolds satisfies a notion of excision analogous to that introduced in [AFT16] for stratified spaces.

Contents

1. Introduction

We begin with a high-level explanation of factorization homology as it applies to smooth or topological manifolds. We then introduce orbifolds, and explain the main results of this thesis: an extension of factorization homology to orbifolds, and a proof of its main property: excision. We conclude the introduction by explaining some limitations of our approach, proposing future directions of study, and by surveying related work.

1.1. **Factorization homology of manifolds.** The underlying topological space of a smooth manifold is locally homeomorphic to Euclidean space $\mathbb{R}^n$. The goal of *factorization homology* of manifolds is to exploit this *local* structure to establish *global* invariants of manifolds. Factorization homology was introduced by Lurie [Lur] as *topological chiral homology* and developed by Ayala, Francis and Tanaka for topological manifolds [AF15] and for stratified spaces [AFT16], [AFT17].

The input of the theory is a *disk algebra* A with values in a symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$.[1] Loosely speaking, A specifies a choice of object $A(\mathbb{R}^n)$ of $\mathcal{V}^{\otimes}$ for Euclidean n-space, and for each open embedding of finite disjoint unions $\sqcup_i \mathbb{R}^n \hookrightarrow \sqcup_j \mathbb{R}^n$, it specifies a map

$$\bigotimes_i A(\mathbb{R}^n) \to \bigotimes_j A(\mathbb{R}^n)$$

in $\mathcal{V}^{\otimes}$. Furthermore, A is required to be *locally constant*: the open embeddings $\mathbb{R}^n \hookrightarrow \mathbb{R}^n$ are required to be sent to equivalences in $\mathcal{V}$. This can be summarised as follows: the input $A\colon \mathbf{Disk}_n \to \mathcal{V}^{\otimes}$ is a *locally constant algebra* of symmetric monoidal ∞-categories, where $\mathbf{Disk}_n$ has objects finite disjoint unions of $\mathbb{R}^n$, as

[1] The symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ is required to be $\otimes$-presentable, which roughly means it is presentable and its tensor product commutes with colimits in each variable.

maps open embeddings, and obtains a symmetric monoidal structure from disjoint union. The output of factorization homology on a manifold M, suggestively denoted

$$\int_M A \quad \in \quad \mathcal{V}^{\otimes},$$

is to be understood as a sort of *gluing* of the algebraic information provided by applying A to the open disks openly embedded in M. When $\mathcal{V}^{\otimes}$ is chain complexes with direct sum, factorization homology recovers ordinary homology. For coefficients in spectra, factorization homology recovers generalised homology.

We can more precisely define factorization homology as a symmetric monoidal left Kan extension of A along the inclusion $\mathbf{Disk}_n \hookrightarrow \mathbf{Man}_n^{oe}$. Here $\mathbf{Man}_n^{oe}$ is the symmetric monoidal ∞-category of n-manifolds and open embeddings with disjoint union. The pointwise formula for Kan extensions makes formal the intuition that factorization homology glues the values of A on open disks in M:

$$\int_M A \simeq \operatorname{colim}(\mathbf{Disk}_{n/M} \to \mathbf{Disk}_n \xrightarrow{A} \mathcal{V}^{\otimes}).$$

Instead of starting with a locally constant disk algebra $A\colon \mathbf{Disk}_n \to \mathcal{V}^{\otimes}$, Ayala and Francis set up a symmetric monoidal ∞-category $\mathcal{D}\mathbf{isk}_n$ which has as objects the finite disjoint unions of $\mathbb{R}^n$, as morphisms the open embeddings, and as higher morphisms the (higher) isotopies of open embeddings. They then prove that disk algebras $A'\colon \mathcal{D}\mathbf{isk}_n \to \mathcal{V}^{\otimes}$ are equivalent to locally constant disk algebras $A\colon \mathbf{Disk}_n \to \mathcal{V}^{\otimes}$, [AF15, Remark 2.20]. Factorization homology of A' can then be defined as a symmetric monoidal left Kan extension of A' along $\mathcal{D}\mathbf{isk}_n \hookrightarrow \mathcal{M}\mathbf{an}_n$, where $\mathcal{M}\mathbf{an}_n$ also has isotopies of open embeddings as higher morphisms. Setting up factorization homology from this point of view is crucial in the proofs of excision given in [AF15], [AFT16].

1.2. **Excision for manifolds.** One of the main properties of factorization homology of manifolds, which is the main justification for calling it a *homology* theory, is a version of *excision*. To state it, we need the notion of a *collar-gluing* of a manifold M. This is a map $f\colon M \to I := [-1,1]$ that is a trivial fiber bundle with fiber M_0 when restricted to the interior $(-1,1) \subset I$. For such a decomposition of M, we denote by M_- the open submanifold $f^{-1}[-1,1)$ and by M_+ the open submanifold $f^{-1}(-1,1]$.

Given a symmetric monoidal functor $F\colon \mathcal{M}\mathbf{an}_n \to \mathcal{V}^{\otimes}$, the object $F(M_0 \times (-1,1))$ can be viewed as an associative algebra object in $\mathcal{V}^{\otimes}$ by transferring the gluing map

$$(M_0 \times (-1,1)) \sqcup (M_0 \times (-1,1)) \to M_0 \times (-1,1)$$

along F. Similarly, $F(M_-)$ and $F(M_+)$ can be viewed as right and left modules over $F(M_0 \times (-1,1))$, respectively. We say that F *satisfies excision* if for any collar-gluing $M \to I$, the value of F on M is the relative tensor product

$$F(M) \simeq F(M_-) \bigotimes_{F(M \times (-1,1))} F(M_+).$$

In [AF15, Lemma 3.18], Ayala and Francis prove that factorization homology of topological manifolds satisfies excision for any choice of disk algebra.

1.3. **Orbifolds.** The goal of this thesis is to define factorization homology for *orbifolds*, which are a generalisation of manifolds. Instead of spaces that locally look like Euclidean space, orbifolds locally look like Euclidean spaces equipped with linear actions of finite groups. A first approach to representing these 'local building blocks' would be to consider the *quotient space* $\mathbb{R}^n/G$ of a finite group G acting linearly on $\mathbb{R}^n$. This representation is quite coarse: points where the isotropy subgroups are non-trivial (i.e. points which are fixed by some $g \neq 1 \in G$) are not distinguished from points where the action is free.

For an example, consider the action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{R}^2$ given by rotation by $2\pi/n$. The quotient space is homeomorphic to $\mathbb{R}^2$, but the projection $\mathbb{R}^2 \twoheadrightarrow \mathbb{R}^2/(\mathbb{Z}/n\mathbb{Z})$ is not naturally a smooth map. The problem is at $0 \in \mathbb{R}^2$, which has non-trivial isotropy subgroup: it is fixed by every $k \in \mathbb{Z}/n\mathbb{Z}$. This yields a *singularity* of order n in the quotient space, which can be thought of as a cone with vertex 0.

To avoid throwing away the information encoded by non-trivial isotropy subgroups, we can consider the *action groupoid* $G \ltimes \mathbb{R}^n$ of a linear action of G on $\mathbb{R}^n$, where the source and target maps $G \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are given by projection to $\mathbb{R}^n$ and application of the group action, respectively. We can think of an orbifold as a Lie groupoid that 'locally looks' like $G \ltimes \mathbb{R}^n$. It turns out that the *proper étale Lie groupoids* are of this form (we make this precise in Proposition 2.27), and we thus define *orbifold groupoids* to be proper étale Lie groupoids. This point of view was developed by [MP], we refer the reader there for more details.

There is one more step in setting up the theory of orbifolds: producing a category. Viewing orbifolds as groupoids, this should be a 2-category. It turns out that the 2-category of Lie groupoids and smooth functors is not well-behaved: fully faithful and essentially surjective smooth functors may not have smooth inverses. These can be inverted to produce a category in which they are equivalences; this is set up in [Pro96]. We instead consider the *classifying stack* $\mathbf{B}\mathcal{G}$ associated to a Lie groupoid $\mathcal{G}$. This is a space-valued sheaf on the category of smooth manifolds **Man**. It turns out that the fully faithful and essentially surjective smooth functors become equivalences when passing to classifying stacks, so we can view orbifolds as those stacks that are equivalent to classifying stacks of proper étale Lie groupoids in the ∞-category of stacks $\mathrm{Shv}(\mathbf{Man})$. This is proven in [Pro96], see also [Ler09]. We discuss viewing orbifolds as stacks in Section 2.3.

1.4. **Factorization homology of orbifolds.** In the ∞-category $\mathrm{Shv}(\mathbf{Man})$ there is a notion of maps that are open embeddings (see Section 2), we call these *open substacks*. We thus obtain the ∞-subcategories $\mathbf{ODisk}_n^{oe}$ and $\mathbf{Orb}_n^{oe}$ spanned by the orbifold n-disks $[\mathbb{R}^n/G]$ (coming from the action groupoids $G \ltimes \mathbb{R}^n$) and by the n-orbifolds, respectively, with as maps the open substacks. Our first step toward defining factorization homology for orbifolds is to construct new ∞-categories which contain, in the higher morphisms, applicable notions of isotopies of open substacks. We carry out this construction in Section 5, yielding ∞-categories $\mathcal{OD}\mathbf{isk}_n$ and $\mathcal{O}\mathbf{rb}_n$ of orbifold n-disks and n-orbifolds, respectively. Our approach is via the *enriched* ∞*-categories* developed in [GH15] (see Section 3). The idea is to first set up a category of orbifolds *enriched in stacks*: we define an *embedding stack* $\mathrm{Emb}(\mathcal{M}, \mathcal{N})$ between any two orbifolds that contains the open substacks and (higher) isotopies between them. Next, we apply a *shape functor* $\Pi_\infty \colon \mathrm{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$ to these embedding stacks. We can think of Π_∞ as computing the *homotopy type* of a stack.

This yields a *space* $\Pi_\infty \operatorname{Emb}(\mathcal{M}, \mathcal{N})$ of morphisms between any two orbifolds, and thus ordinary ∞-categories $\mathcal{OD}\mathbf{isk}_n$ and $\mathcal{O}\mathbf{rb}_n$.

In Section 6 we endow the ∞-categories $\mathcal{OD}\mathbf{isk}_n$ and $\mathcal{O}\mathbf{rb}_n$ with symmetric monoidal structures coming from the coproduct in $\operatorname{Shv}(\mathbf{Man})$. Given an algebra $A\colon \mathcal{OD}\mathbf{isk}_n \to \mathcal{V}^{\otimes}$, we can then define factorization homology of orbifolds as a left Kan extension along $\mathcal{OD}\mathbf{isk}_n \hookrightarrow \mathcal{O}\mathbf{rb}_n$, recovering the definition for manifolds as a special case. For an orbifold $\mathcal{M}$, we obtain the formula

$$\int_{\mathcal{M}} A \simeq \operatorname{colim}(\mathcal{OD}\mathbf{isk}_{n/\mathcal{M}} \to \mathcal{OD}\mathbf{isk} \xrightarrow{A} \mathcal{V}^{\otimes}).$$

In Section 7, we show that this left Kan extension defines a symmetric monoidal functor $\mathcal{O}\mathbf{rb}_n \to \mathcal{V}^{\otimes}$.

1.5. **Excision for orbifolds.** Our main result is that factorization homology of orbifolds satisfies a version of excision analogous to that for manifolds. Via the restriction map $\operatorname{Shv}(\mathbf{Man}^{\partial}) \to \operatorname{Shv}(\mathbf{Man})$ from stacks on manifolds with boundary to stacks on manifolds, we can view the closed interval I as a stack on $\mathbf{Man}$. We define an *orbifold collar-gluing* to be a map $f\colon \mathcal{M} \to I \in \operatorname{Shv}(\mathbf{Man})$ that restricts to the projection $\mathcal{M}_0 \times (-1,1) \to (-1,1)$ when pulling back along the inclusion $(-1,1) \hookrightarrow I$. We show in Section 8 that given a collar gluing f, the factorization homology of $\mathcal{M}$ decomposes as a relative tensor product

$$\int_{\mathcal{M}} A \simeq \int_{\mathcal{M}_-} A \bigotimes_{\int_{\mathcal{M}_0 \times (-1,1)} A} \int_{\mathcal{M}_+} A,$$

where $\mathcal{M}_- = f^{-1}[-1,1)$ and $\mathcal{M}_+ = f^{-1}(-1,1]$. Our approach adapts the proof of excision given for smooth manifolds and stratified spaces in [AFT16]. In their proof, Ayala and Francis use *configuration spaces* as topological models of the mapping spaces $\operatorname{Emb}(\sqcup_i \mathbb{R}^n, M)$ of open embeddings. The intuitive idea is that when working up to isotopy, one can shrink a disk to a point without losing information about how this disk is embedded in a manifold. In the case of orbifolds, we do not have access to such topological models, but the intuition is unchanged: we should still be able to shrink an open substack $[\mathbb{R}^n/G] \to \mathcal{M}$ without changing it up to isotopy. We make use of this idea in Proposition 8.5, which is the main place in which our proof of excision differs from that of Ayala and Francis.

1.6. **Related work.** It is conjectured in [AFT17, Conjecture 1.5.3] that orbifolds can be viewed as stratified spaces. To our knowledge this remains unproven, and we are not sure whether giving a stratification of an orbifold and applying the factorization homology of [AFT16] agrees with our notion of factorization homology of orbifolds.

Factorization homology has been developed for manifolds equipped with a G-action, under the name of *genuine equivariant factorization homology*, in [Hor19], and as *factorization homology of global quotient orbifolds* in [Wee18]. The papers differ in setup, but both restrict to a manifold on which a group acts globally, as opposed to considering general orbifolds, which are defined by groups acting locally.

1.7. **Limitations and future work.** In this work, we do not consider an orbifold analogue of a *framing* of a manifold, nor how it interacts with factorization homology. This is one of the main ways Ayala and Francis do computations for factorization homology of manifolds, so this limits our ability to translate their computations to the orbifold case.

In both [AF15] and in [AFT16], Ayala and Francis show that the symmetric monoidal functors $\mathcal{M}\mathbf{an}_n \to \mathcal{V}^{\otimes}$ that satisfy excision come, up to equivalence, from factorization homology of disk algebras. We suspect a similar result might hold for symmetric monoidal functors $\mathcal{O}\mathbf{rb}_n \to \mathcal{V}^{\otimes}$. Ayala and Francis also derive a *non-abelian* version of Poincaré duality ([AF15, Theorem 3.24]). Extending this result to orbifolds was the original motivation for this work, but it depends on characterizing symmetric monoidal functors satisfying excision as obtained from factorization homology, so we do not attempt to set this up here.

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2. Stacks, differentiable stacks, and orbifolds

In this section we introduce the standard definitions of stacks on differentiable manifolds and explain how orbifolds can be defined in this context. We closely follow Bastiaan Cnossen's PhD thesis [Cno23, Chapter II.2], in which the theory of differentiable stacks is developed directly in the context of space-valued sheaves. We summarize the definitions and results we will need in this section in order to keep this work self-contained.

2.1. Stacks on smooth manifolds.

Definition 2.1. We will denote by $\mathbf{Man}$ the 1-category of smooth Hausdorff manifolds with smooth maps between them. For simplicity, we will assume that the dimension of each connected component in a manifold is the same.[2] We put the Grothendieck topology on $\mathbf{Man}$ in which a sieve on a manifold is a covering sieve iff it contains a covering consisting of jointly surjective open embeddings.

Definition 2.2. Let $\mathcal{U}$ be an open cover of a smooth manifold T by manifolds. Consider the disjoint union $\mathbf{U} := \coprod_{U\in\mathcal{U}} U$. The *Čech nerve* of $\mathcal{U}$ is the simplicial manifold

$$\mathcal{U}_\bullet := (\dots \mathbf{U} \times_T \mathbf{U} \times_T \mathbf{U} \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathbf{U} \times_T \mathbf{U} \rightrightarrows \mathbf{U})$$

that at degree n is given by the $(n+1)$th fiber product of $\mathbf{U}$ over T with itself.

Definition 2.3. A presheaf $F \in \mathrm{Fun}(\mathbf{Man}^{\mathrm{op}}, \boldsymbol{\mathcal{S}})$ is called a *sheaf* if for each smooth manifold T and each open cover $\mathcal{U}$ of T, the map $F(T) \to \lim F(\mathcal{U}_\bullet)$ is an equivalence.

Definition 2.4. The ∞-category of *stacks*, denoted $\mathrm{Shv}(\mathbf{Man})$, is the full subcategory of presheaves $\mathrm{Fun}(\mathbf{Man}^{\mathrm{op}}, \boldsymbol{\mathcal{S}})$ spanned by the sheaves. We will refer to an object $\mathcal{X} \in \mathrm{Shv}(\mathbf{Man})$ as a *stack on smooth manifolds*, or just as a *stack*.

[2] This differs from [Cno23, Definition 2.1.1], where the dimension is allowed to vary across connected components.

Remark 2.5. The ∞-category $\mathrm{Shv}(\mathbf{Man})$ is an ∞-*topos*. In particular, it is Cartesian closed, with the mapping stacks $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ defined on a test manifold T by $\mathrm{Map}(\mathcal{X}, \mathcal{Y})(T) := \mathrm{Map}_{\boldsymbol{\mathcal{S}}}(\mathcal{X} \times T, \mathcal{Y})$.

Remark 2.6. Open covers on **Man** define a subcanonical topology, so that by the Yoneda embedding, we can view any smooth manifold as a stack. We will use the notation $M \in \mathrm{Shv}(\mathbf{Man})$ for M a smooth manifold. The Yoneda embedding preserves finite limits, and by descent it preserves arbitrary coproducts.

Remark 2.7. The category of smooth manifolds **Man** does not have all pullbacks. However, when one of the maps in a pullback diagram is a *submersion*, then the pullback does exist ([Cno23, Proposition C.9]). This motivates the following definition of representability of maps in $\mathrm{Shv}(\mathbf{Man})$.

Definition 2.8 ([Cno23, Definition 2.1.4])**.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map of stacks on smooth manifolds.

(1) The map f is called a *representable submersion* if for each map $Y \to \mathcal{Y}$ from a smooth manifold Y, the pullback $\mathcal{X} \times_{\mathcal{Y}} Y$ is a smooth manifold and the induced map $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is a submersion of smooth manifolds. If each induced map $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is also surjective, then we call f a *representable surjective submersion*.
(2) We say f is *representable* if for each smooth manifold Y and each representable submersion $Y \to \mathcal{Y}$, the pullback $\mathcal{X} \times_{\mathcal{Y}} Y$ is a smooth manifold.

Notice that a map of smooth manifolds is representable as a map of stacks, and it is a representable (surjective) submersion of stacks if and only if it is a (surjective) submersion of manifolds.

2.2. **Differentiable stacks.** The notion of stacks developed in the previous section is quite general. We will now consider the class of *differentiable stacks*, which can be thought of as sorts of 'quotients' of manifolds, and as such share many desirable properties with manifolds themselves. It makes sense that our notion of a 'manifold quotient' should be more general than a manifold: the quotient space of a manifold by the action of a Lie groupoid is in general not a manifold.

Definition 2.9 ([Cno23, Definition 2.2.1])**.** Let $\mathcal{X}$ be a stack on smooth manifolds. A *representable atlas* on $\mathcal{X}$ is a representable surjective submersion $X \to \mathcal{X}$ from a smooth manifold X. A stack that admits a representable atlas is called a *differentiable stack*.

2.2.1. *Local properties of maps.*

Definition 2.10. A map $f\colon X \to Y$ of smooth manifolds is an *open embedding* if it is both an injective immersion and an open map. It is a *closed embedding* if it is both an injective immersion and a closed map.

We have the following two closely-related definitions in the case of stacks.

Definition 2.11. A representable submersion of stacks $f\colon \mathcal{U} \to \mathcal{X}$ *exhibits* $\mathcal{U}$ *as an open substack of* $\mathcal{X}$ if, for every smooth manifold M and every map $M \to \mathcal{X}$, the pullback $M \times_{\mathcal{X}} \mathcal{U} \to M$ is an open embedding of smooth manifolds.

Replacing open with closed in the above definition gives a *closed substack*.

Definition 2.12. A representable morphism of stacks $f\colon \mathcal{U} \to \mathcal{X}$ is a *representable open embedding* if for every representable submersion $X \to \mathcal{X}$, the map $\mathcal{U} \times_{\mathcal{X}} X \to X$ is an open embedding of smooth manifolds.

Remark 2.13. Representable submersions and open substacks are closed under arbitrary pullbacks. That is, if $\mathcal{X} \to \mathcal{Y}$ is a representable submersion (open substack) and $\mathcal{Y}' \to \mathcal{Y}$ is any morphism of stacks, then the map $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ is a representable submersion (open substack).

The potential for confusion from the similar-looking Definition 2.11 and Definition 2.12 is alleviated in the case of differentiable stacks by the following proposition.

Proposition 2.14 ([Cno23, Lemma 2.4.4])**.** *Let $f\colon \mathcal{U} \to \mathcal{X}$ be a map of differentiable stacks. The following are equivalent:*

(1) The map f exhibits $\mathcal{U}$ as an open substack of $\mathcal{X}$.
(2) The map f is a representable open embedding.

Proposition 2.15 ([Cno23, Lemma 2.4.3])**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map of differentiable stacks. The following are equivalent:*

(1) The map f is a representable submersion of stacks.
(2) For each representable submersion $Y \to \mathcal{Y}$ from a manifold Y, the map $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is a smooth submersion of manifolds.

Definition 2.16. A representable morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is called *proper* if for every representable submersion $Y \to \mathcal{Y}$ the induced map $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is a proper map.

Definition 2.17. A representable morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is called a *local diffeomorphism* if for every representable submersion $Y \to \mathcal{Y}$ the induced map $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is a local diffeomorphism of manifolds.

2.2.2. *Lie groupoids give rise to differentiable stacks.* We have described a differentiable stack as being a sort of quotient of a manifold X, in the sense that we have a representable surjective submersion $X \to \mathcal{X}$. Another way to model such quotients is via Lie groupoids. These two perspectives agree, as we will recall in this section.

Definition 2.18. A Lie groupoid $\mathcal{G} := (G_1 \rightrightarrows G_0)$ is a groupoid such that both G_1 and G_0 are smooth manifolds, all structure maps are smooth, and the source and target maps are surjective smooth submersions.

From any Lie groupoid we can construct a differentiable stack, and indeed all differentiable stacks can be constructed in this way up to equivalence of stacks.

Definition 2.19 ([Cno23, Definition 2.3.1])**.** The *nerve* of a Lie groupoid $\mathcal{G} = (G_1 \rightrightarrows G_0)$ is the simplicial object $N(\mathcal{G})\colon \Delta^{\mathrm{op}} \to \mathbf{Man}$ given by iterated pullbacks of G_1 over G_0:

$$\cdots \qquad G_1 \times_{G_0} G_1 \times_{G_0} G_1 \; \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \; G_1 \times_{G_0} G_1 \; \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \; G_1 \rightrightarrows G_0.$$

Definition 2.20 ([Cno23, Definition 2.3.2])**.** Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a Lie groupoid. Its *classifying stack* $\mathbf{B}\mathcal{G} \in \mathrm{Shv}(\mathbf{Man})$ is the colimit of the diagram

$$\Delta^{\mathrm{op}} \xrightarrow{N(\mathcal{G})} \mathbf{Man} \to \mathrm{Shv}(\mathbf{Man}).$$

Lemma 2.21 ([Cno23, Lemma 2.3.3])**.** *Let $\mathcal{G}$ be a Lie groupoid. The map $G_0 \to \mathbf{B}\mathcal{G}$ is a representable atlas for $\mathbf{B}\mathcal{G}$, so that $\mathbf{B}\mathcal{G}$ is a differentiable stack.*

Proposition 2.22 ([Cno23, Corollary 2.3.8])**.** *Every differentiable stack* $\mathcal{X} \in \mathrm{Shv}(\mathbf{Man})$ *is equivalent to the classifying stack* $\mathbf{B}\mathcal{G}$ *of a Lie groupoid* $\mathcal{G}$*.*

Example 2.23 ([Cno23, Example 2.3.5])**.** Let G be a Lie group acting smoothly on a manifold $M \in \mathbf{Man}$. The source and target $G \times M \rightrightarrows M$ given by $s\colon (g,m) \mapsto m$ and $t\colon (g,m) \mapsto g \cdot m$ define a Lie groupoid $G \ltimes M$, called the *action groupoid* associated to the action of G on M. We will denote the classifying stack $\mathbf{B}(G \ltimes M)$ by $[M/G]$.

The following lemma will be central to our proof of excision for factorization homology on orbifolds, c.f. Proposition 8.5.

Lemma 2.24. *Let* G *be a Lie group acting smoothly on* $U, M \in \mathbf{Man}$*. Let* $\phi\colon U \hookrightarrow M$ *be a* G*-equivariant open embedding of manifolds. The induced inclusion of action groupoids* $G \ltimes U \hookrightarrow G \ltimes M$ *induces a map of differentiable stacks* $[U/G] \hookrightarrow [M/G]$ *that exhibits* $[U/G]$ *as an open substack of* $[M/G]$*.*

Proof. This is a special case of [Cno23, Lemma 3.6.4], because $G \ltimes U$ is isomorphic to the pullback groupoid $\phi^*(G \ltimes M)$ defined in [Cno23, Example D.7]. □

2.3. **Orbifolds as separated étale stacks.** Manifolds are topological spaces that '*look locally like Euclidean space*'. We will now consider a class of differentiable stacks that we will think of as '*looking locally like quotients of Euclidean space*' by linear actions of finite groups. These will be what we will call *orbifolds*. To build some intuition for this perspective, we will start by considering *Lie groupoids* that look locally like quotients of Euclidean spaces.

Definition 2.25. A Lie groupoid $\mathcal{G} := (G_1 \rightrightarrows G_0)$ is called *proper* if the map $(s,t)\colon G_1 \to G_0 \times G_0$ is a proper map. A Lie groupoid $\mathcal{G}$ is called *étale* if the source and target maps s and t are étale maps, i.e. local diffeomorphisms.

Definition 2.26 ([Ler09, Definition 2.21])**.** Given a Lie groupoid $\mathcal{G} := (G1 \rightrightarrows G_0)$ and an open set $U \subset G_0$, the space $s^{-1}(U) \cap t^{-1}(U)$ is an open submanifold of G_1, and restriction to this space of arrows gives a Lie groupoid with space of objects U. This is the *restriction of* $\mathcal{G}$ *to* U and is denoted $\mathcal{G}|_U$.

The following proposition says that proper étale Lie groupoids are locally quotients of Euclidean space by linear actions of finite groups.

Proposition 2.27 ([Ler09, Proposition 2.23])**.** *Let* G *be a proper étale Lie groupoid. For any point* $x \in G_0$ *there is an open neighborhood* U_x *of* x *so that the restriction* $G|_U$ *is isomorphic to an action groupoid* $G_x \times U_x \rightrightarrows U_x$*, for* G_x *a finite group*[3]*. Moreover, we may take* U *to be an open ball in Euclidean space centered at* 0 *and the action of* G_x *to be linear.*

Remark 2.28. At this point, the reader might wonder why we do not define orbifolds to be proper étale Lie groupoids and work in the 2-category of Lie groupoids with smooth functors. The reason is that in this category, fully faithful and essentially surjective smooth functors are not necessarily invertible by smooth functors. One fix is to invert these functors, which are called *Morita equivalences*. This is standard in the orbifold literature, see [Pro96], [MP]. It turns out that applying the classifying stack of Proposition 2.22 and working with the resulting differentiable

[3] This means there is a smooth functor $f\colon \mathcal{G}|_{U_x} \to G_x \ltimes U_x$ with smooth inverse.

stacks in Shv(**Man**) sends Morita equivalences to equivalences of stacks and thus accomplishes the same goal. For more on the relationship between Lie groupoids and differentiable stacks, see for example [BX08], in particular Theorem 2.26, or [Ler09].

We could now apply Proposition 2.22 and define an orbifold to be a differentiable stack that is equivalent to $\mathbf{B}\mathcal{G}$ for $\mathcal{G}$ a proper étale Lie groupoid. We choose to give a definition via properties of differentiable stacks; Proposition 2.31 says both approaches coincide.

Definition 2.29 ([Cno23, Definition 3.3.1])**.** A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is called *separated* if its diagonal

$$\Delta_f\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

is a proper morphism of stacks. A stack is *separated* if the map $\mathcal{X} \to *$ is separated.

Definition 2.30 ([Cno23, Definition 3.3.4])**.** A differentiable stack $\mathcal{M}$ is called *étale* if it admits a representable atlas $M \to \mathcal{M}$ that is a local diffeomorphism. An *orbifold* is a separated étale stack.

Proposition 2.31 ([Cno23, Example 3.3.5])**.** *The classifying stack of a proper étale Lie groupoid is an orbifold in the sense of Definition 2.30; conversely, every orbifold is equivalent to the classifying stack of a proper étale Lie groupoid.*

Remark 2.32. Let $\mathcal{M}$ be an étale stack. Then any two representable atlases $M, M' \rightrightarrows \mathcal{M}$ are manifolds of the same dimension, since the projection maps $M \times_{\mathcal{M}} M' \to M'$ and $M \times_{\mathcal{M}} M' \to M$ are local homeomorphisms. We can thus define the *dimension* of an étale stack as the manifold dimension of a choice of atlas.

Definition 2.33. We let $\mathbf{Orb}_n$ denote the full subcategory of Shv(**Man**) spanned by the orbifolds of dimension n.

The following proposition, a counterpart to Proposition 2.27, recovers the intuition that an orbifold should '*locally look like a quotient of $\mathbb{R}^n$ by the action of a finite group.*'

Proposition 2.34. *Let $\mathcal{M}$ be an orbifold of dimension n. There exists a collection of finite groups G_i, each acting linearly on $\mathbb{R}^n$, and corresponding open substacks $[\mathbb{R}^n/G_i] \hookrightarrow \mathcal{M}$, such that the induced map*

$$\coprod_i [\mathbb{R}^n/G_i] \to \mathcal{M}$$

is an effective epimorphism.

Proof. From [Cno23, Theorem 3.7.2] we obtain a collection of open substacks $[\mathbb{R}^n/G_i] \hookrightarrow \mathcal{M}$. The induced map $\rho\colon \coprod_i [\mathbb{R}^n/G_i] \to \mathcal{M}$ is a representable submersion by universality of colimits in Shv(**Man**): for any test manifold $Y \to \mathcal{M}$, the induced map $\coprod_i [\mathbb{R}^n/G_i] \times_{\mathcal{M}} Y \to Y$ is a submersion of manifolds because each map $[\mathbb{R}^n/G_i] \times_{\mathcal{M}} Y \to Y$ is an open embedding of manifolds and thus a submersion of manifolds. By [Cno23, Remark 2.1.6], the representable submersion ρ is an effective epimorphism if and only if it is a representable *surjective* submersion. So let $y \in Y$ be a point. We get a map $\{y\} \to Y \to \mathcal{M}$, and by [Cno23, Theorem

3.7.2], there is a corresponding diagram

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \{y\} \arrow[d] \\
{[\mathbb{R}^n/G_i] \times_{\mathcal{M}} Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\
{[\mathbb{R}^n/G_i]} \arrow[r, hook] & \mathcal{M},
\end{tikzcd}$$

where $\{0\} \to [\mathbb{R}^n/G_i]$ is the point corresponding to $0 \in \mathbb{R}^n$. □

Definition 2.35. An *orbifold n-disk* is a stack equivalent to $[\mathbb{R}^n/G]$, for G a finite group acting linearly on $\mathbb{R}^n$. We let $\mathbf{ODisk}_n$ denote the full subcategory of $\mathbf{Orb}_n$ spanned by the finite disjoint unions of orbifold n-disks.

3. Enriched ∞-categories

In this section, we recall the setup of *enriched ∞-categories* developed by Gepner and Haugseng in [GH15]. Our exposition closely follows theirs, and we do not reproduce proofs. We aim instead only to state the definitions and results we will need later on to construct enriched ∞-categories of stacks and orbifolds and to produce from these honest ∞-categories via change of enrichment.

3.1. Non-symmetric ∞-operads.

Definition 3.1 ([GH15, Definition 2.2.3])**.** A morphism $\phi\colon [n] \to [m]$ in Δ is *inert* if $\phi(i) = \phi(0) + i$ for each $i = 0, \dots, n$, and *active* if it preserves the endpoints, i.e. if $\phi(0) = 0$ and $\phi(n) = m$. We denote the inert morphism $[1] \to [n]$ given by the inclusion $i-1, i \hookrightarrow [n]$ by ρ_i for $i = 1, \dots, n$.

Definition 3.2 ([GH15, Definition 2.2.6])**.** A *non-symmetric ∞-operad* is an inner fibration $\pi\colon \mathcal{O} \to \Delta^{\mathrm{op}}$ such that:

(1) For every inert morphism $\phi\colon [m] \to [n]$ in Δ^{op} and every $\mathcal{X} \in \mathcal{O}_{[n]}$ there is a π-coCartesian morphism $\mathcal{X} \to \phi_!\mathcal{X}$ over ϕ.
(2) For every $[n] \in \Delta^{\mathrm{op}}$ the functor $\mathcal{O}_{[n]} \to (\mathcal{O}_{[1]})^{\times n}$ induced by the π-coCartesian arrows over the inert maps ρ_i is an equivalence of ∞-categories.
(3) For every $\phi : [n] \to [m]$ in Δ^{op} and $\mathcal{Y} \in \mathcal{O}_{[m]}$, composition with the coCartesian arrows $\mathcal{Y} \to \mathcal{Y}_i$ over the inert morphisms ρ_i gives an equivalence
$$\mathrm{Map}^{\phi}_{\mathcal{O}}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\simeq} \prod_i \mathrm{Map}^{\rho_i \circ \phi}_{\mathcal{O}}(\mathcal{X}, \mathcal{Y}_i),$$
where $\mathrm{Map}^{\phi}_{\mathcal{O}}(\mathcal{X}, \mathcal{Y})$ is the subspace of $\mathrm{Map}_{\mathcal{O}}(\mathcal{X}, \mathcal{Y})$ of maps above $\phi \in \Delta^{\mathrm{op}}$.

Definition 3.3 ([GH15, Definition 2.2.10])**.** A morphism of non-symmetric ∞-operads $\mathcal{O}$ to $\mathcal{P}$ is a commutative diagram

$$\begin{tikzcd}
\mathcal{O} \arrow[rr, "f"] \arrow[dr] & & \mathcal{P} \arrow[dl] \\
& \Delta^{\mathrm{op}} &
\end{tikzcd}$$

such that f carries coCartesian morphisms in $\mathcal{O}$ that map to inert morphisms in Δ^{op} to coCartesian morphisms in $\mathcal{P}$. We will also call such a morphism $\mathcal{O} \to \mathcal{P}$

an $\mathcal{O}$*-algebra in* $\mathcal{P}$. We get an ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ of $\mathcal{O}$-algebras in $\mathcal{P}$ by taking the full subcategory of $\mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{P})$ spanned by the morphisms of non-symmetric operads.

Definition 3.4 ([GH15, Definition 2.2.13])**.** A *monoidal* ∞*-category* is a non-symmetric ∞-operad $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ that is also a coCartesian fibration.

Definition 3.5 ([GH15, Definition 2.2.14])**.** For $\mathcal{V}^{\otimes}$ and $\mathcal{W}^{\otimes}$ monoidal ∞-categories, a map of non-symmetric operads $\mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ is called a *lax monoidal functor.* A lax monoidal functor that preserves *all* coCartesian morphisms is a *monoidal functor.*

Remark 3.6. By [GH15, Remark 3.1.5], expanded upon in [GH15, Section 3.2], there is a natural map $c\colon \Delta^{\mathrm{op}} \to \mathbf{Fin}_*$ that can be used to pull back symmetric ∞-operads and symmetric monoidal ∞-categories to non-symmetric ∞-operads and monoidal ∞-categories, respectively.

Definition 3.7 ([GH15, Definition 3.2.7])**.** There is an ∞-category $\mathrm{Opd}_{\infty}^{\mathrm{ns}}$ of non-symmetric ∞-operads. Given two non-symmetric ∞-operads $\mathcal{O}$ and $\mathcal{P}$, the space of maps $\mathrm{Map}_{\mathrm{Opd}_{\infty}^{\mathrm{ns}}}(\mathcal{O}, \mathcal{P})$ is the subspace of $\mathrm{Map}_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{P})$ given by the components corresponding to inert-morphism-preserving maps.

Definition 3.8 ([GH15, Definition 3.6.1])**.** Let $\mathcal{O}$ be a non-symmetric ∞-operad. There is a functor $(\mathrm{Opd}_{\infty}^{\mathrm{ns}})^{\mathrm{op}} \to \mathbf{Cat}_{\infty}$ sending a non-symmetric ∞-operad $\mathcal{P}$ to $\mathrm{Alg}_{\mathcal{P}}(\mathcal{O})$. We define

$$\mathrm{Alg}(\mathcal{O}) \to \mathrm{Opd}_{\infty}^{\mathrm{ns}}$$

to be a Cartesian fibration corresponding to this functor.

3.2. **Generalised non-symmetric ∞-operads.**

Definition 3.9 ([GH15, Definition 2.3.1])**.** We let $\Delta_{\mathrm{int}}^{\mathrm{op}}$ be the subcategory of Δ^{op} spanned by the inert morphisms. We denote by $\mathcal{G}^{\Delta} \subset \Delta_{\mathrm{int}}^{\mathrm{op}}$ the full subcategory spanned by $[0]$ and $[1]$, and by $\mathcal{G}^{\Delta}_{[n]/}$ the pullback $(\Delta_{\mathrm{int}}^{\mathrm{op}})_{[n]/} \times_{\Delta^{\mathrm{op}}} \mathcal{G}^{\Delta}$ consisting of inert maps from $[n]$ to $[0]$ and $[1]$.

Definition 3.10 ([GH15, Definition 2.4.1])**.** A *generalised non-symmetric* ∞*-operad* is an inner fibration $\pi\colon \mathcal{M} \to \Delta^{\mathrm{op}}$ such that:

(1) For every inert morphism $\phi\colon [m] \to [n]$ in Δ^{op} and every $\mathcal{X} \in \mathcal{M}_{[n]}$ there is a π-coCartesian morphism $\mathcal{X} \to \phi_!\mathcal{X}$ over ϕ.

(2) For every $[n] \in \Delta^{\mathrm{op}}$ the functor

$$\mathcal{M}_{[n]} \to \lim_{[n]\to[i]\in\mathcal{G}^{\Delta}_{[n]/}} \mathcal{M}_{[i]} \simeq \mathcal{M}_{[1]} \times_{\mathcal{M}_{[0]}} \cdots \times_{\mathcal{M}_{[0]}} \mathcal{M}_{[1]}$$

induced by the coCartesian maps over the inert maps in $\mathcal{G}^{\Delta}_{[n]/}$ is an equivalence of ∞-categories.

(3) For every morphism $\phi\colon [n] \to [m]$ in Δ^{op} and $\mathcal{Y} \in \mathcal{M}_{[m]}$, composition with coCartesian morphisms $\mathcal{Y} \to \mathcal{Y}_{\alpha}$ over the inert morphisms $\alpha\colon [m] \to [i]$ in $\mathcal{G}^{\Delta}_{[m]/}$ gives an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{M}}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\simeq} \lim_{\alpha\in\mathcal{G}^{\Delta}_{[m]/}} \mathrm{Map}^{\alpha\circ\phi}_{\mathcal{M}}(\mathcal{X}, \mathcal{Y}_{\alpha}),$$

where $\mathrm{Map}^{\phi}_{\mathcal{M}}(\mathcal{X}, \mathcal{Y})$ denotes the subspace of $\mathrm{Map}_{\mathcal{M}}(\mathcal{X}, \mathcal{Y})$ of maps above $\phi \in \Delta^{\mathrm{op}}$.

Definition 3.11 ([GH15, Definition 2.4.2])**.** A morphism of generalised non-symmetric ∞-operads $\mathcal{M}$ to $\mathcal{N}$ is a commutative diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "f"] \arrow[dr] & & \mathcal{N} \arrow[dl] \\ & \Delta^{\mathrm{op}} & \end{tikzcd}$$

such that f carries coCartesian morphisms in $\mathcal{M}$ that map to inert morphisms in Δ^{op} to coCartesian morphisms in $\mathcal{N}$. We will also call such a morphism $\mathcal{M} \to \mathcal{N}$ an *$\mathcal{M}$-algebra in $\mathcal{N}$*.

Definition 3.12 ([GH15, Definition 2.4.3])**.** A *double ∞-category* is a generalised non-symmetric ∞-operad $\mathcal{M} \to \Delta^{\mathrm{op}}$ that is also a coCartesian fibration.

Definition 3.13 ([GH15, Definition 3.2.9])**.** There is an ∞-category $\mathrm{Opd}_\infty^{\mathrm{ns,gen}}$ of generalised non-symmetric ∞-operads. Given two generalised non-symmetric ∞-operads $\mathcal{M}$ and $\mathcal{N}$, the space of maps $\mathrm{Map}_{\mathrm{Opd}_\infty^{\mathrm{ns,gen}}}(\mathcal{M}, \mathcal{N})$ is the subspace of $\mathrm{Map}_{/\Delta^{\mathrm{op}}}(\mathcal{M}, \mathcal{N})$ given by the components corresponding to inert-morphism-preserving maps.

Proposition 3.14 ([GH15, Corollary 3.2.14])**.** *The inclusion* $\mathrm{Opd}_\infty^{\mathrm{ns}} \to \mathrm{Opd}_\infty^{\mathrm{ns,gen}}$ *has a left adjoint*

$$L_{\mathrm{gen}} \colon \mathrm{Opd}_\infty^{\mathrm{ns,gen}} \to \mathrm{Opd}_\infty^{\mathrm{ns}}.$$

Definition 3.15 ([GH15, Definition 4.1.1])**.** Let i denote the inclusion $\{[0]\} \to \Delta^{\mathrm{op}}$. Right Kan extensions along i give a functor $i_* \colon \mathbf{Cat}_\infty \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Cat}_\infty)$. If C is an ∞-category, we write $\Delta_C^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ for a coCartesian fibration corresponding to the functor $i_* C$.

Definition 3.16 ([GH15, Remark 4.1.5], [GH15, Remark 4.1.6])**.** There is a fully faithful right adjoint

$$\Delta_{(-)}^{\mathrm{op}} \colon \mathbf{Cat}_\infty \to \mathrm{Opd}_\infty^{\mathrm{ns,gen}}$$

to the functor $\mathrm{Opd}_\infty^{\mathrm{ns,gen}} \to \mathbf{Cat}_\infty$ that sends a generalised non-symmetric ∞-operad $\mathcal{M}$ to its fibre $\mathcal{M}_{[0]}$ at $[0]$.

3.3. **Categorical algebras in monoidal ∞-categories.**

Remark 3.17 ([GH15, Remark 4.1.2])**.** For an ∞-category C, the simplicial category $i_* \colon \Delta^{\mathrm{op}} \to \mathbf{Cat}_\infty$ of Definition 3.15 has at $i_*([n])$ the product $C^{\times n+1}$, face maps the projections, and degeneracies the diagonals.

Lemma 3.18 ([GH15, Lemma 4.1.3])**.** *Let C be an ∞-category. The coCartesian fibration $\Delta_C^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ of Definition 3.15 is a double ∞-category.*

The following description is discussed in the text of [GH15] after Definition 2.3.4 and before Definition 2.4.5.

Remark 3.19. For $S \in \mathbf{Set}$ a discrete space, the double ∞-category $\Delta_S^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ admits an intuitive description: the ∞-category Δ_S^{op} is the nerve of a 1-category. Its objects of Δ_S^{op} are nonempty sequences $(X_0, \dots, X_n)$ of elements $X_i \in S$. There is a unique morphism $(X_0, \dots, X_n) \to (X_{\phi(0)}, \dots, X_{\phi(m)})$ out of each sequence $(X_0, \dots, X_n)$ for each $\phi \colon [m] \to [n]$ in Δ.

In the general case $S \in \boldsymbol{\mathcal{S}}$, Δ^{op}_S is an ∞-category with fibers $(\Delta^{\mathrm{op}}_S)_{[n]} \simeq S^{n+1}$. The objects of Δ^{op}_S are thus the nonempty sequences $(X_0, \dots, X_n)$ of objects $X_i \in S$. Above $\phi\colon [m] \to [n]$ in Δ is the coCartesian morphism

$$(X_0, \dots, X_n) \to (X_{\phi(0)}, \dots, X_{\phi(m)}).$$

A map $([n], (X_0, \dots, X_n)) \to ([m], (Y_0, \dots, Y_m))$ over ϕ is given by a point in the mapping space

$$\mathrm{Map}_{S^{m+1}}((X_{\phi(0)}, \dots, X_{\phi(m)}), (Y_0, \dots, Y_m)) \simeq \prod_{i=0}^{m} \mathrm{Map}_S(X_{\phi(i)}, Y_i).$$

Definition 3.20 ([GH15, Definition 2.4.5])**.** Let $S \in \boldsymbol{\mathcal{S}}$ be a space and $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ a monoidal ∞-category. A *categorical algebra in* $\mathcal{V}$ with *space of objects* S is a Δ^{op}_S-algebra in $\mathcal{V}$.

Definition 3.21 ([GH15, Definition 2.4.9])**.** Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category, and let $\mathcal{C}\colon \Delta^{\mathrm{op}}_S \to \mathcal{V}^{\otimes}$ and $\mathcal{D}\colon \Delta^{\mathrm{op}}_T \to \mathcal{V}^{\otimes}$ be categorical algebras in $\mathcal{V}$, with object spaces $S, T \in \boldsymbol{\mathcal{S}}$. A $\mathcal{V}$-functor from $\mathcal{C}$ to $\mathcal{D}$ consists of a morphism of spaces $f\colon S \to T$ and a natural transformation $\mathcal{C} \to f^*\mathcal{D}$, where $f^*\mathcal{D}$ denotes the composite

$$\Delta^{\mathrm{op}}_S \xrightarrow{\Delta^{\mathrm{op}}_f} \Delta^{\mathrm{op}}_T \xrightarrow{\mathcal{D}} \mathcal{V}^{\otimes}.$$

Definition 3.22 ([GH15, Definition 4.3.1])**.** Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category. The ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ of *categorical algebras in* $\mathcal{V}$ is defined by the pullback square

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \arrow[r] \arrow[d] & \mathrm{Alg}(\mathcal{V}) \arrow[d] \\
\boldsymbol{\mathcal{S}} \arrow[r, "L_{\mathrm{gen}}\Delta^{\mathrm{op}}_{(-)}"'] & \mathrm{Opd}^{\mathrm{ns}}_{\infty}.
\end{tikzcd}$$

Remark 3.23. By [GH15, Remark 4.3.2] there is an equivalence $\mathrm{Alg}_{L_{\mathrm{gen}}\Delta^{\mathrm{op}}_S}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Alg}_{\Delta^{\mathrm{op}}_S}(\mathcal{V})$ for every space S. The objects of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ are thus categorical algebras in $\mathcal{V}$ in the sense of Definition 3.20, and the functors are $\mathcal{V}$-functors in the sense of Definition 3.21.

Remark 3.24. The objects of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ are sometimes called $\mathcal{V}$-enriched ∞-categories in [GH15]. We will reserve this terminology for the objects of the ∞-category $\mathbf{Cat}^{\mathcal{V}}_{\infty}$, which [GH15] construct as a localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ at the *fully faithful and essentially surjective functors*, and which we explain next.

3.4. **Fully faithful and essentially surjective functors.** Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category. The unit of $\mathcal{V}$ determines an essentially unique *associative algebra object* $I_{\mathcal{V}}\colon \Delta^{\mathrm{op}} \to \mathcal{V}^{\otimes}$ ([GH15, Proposition 3.1.18]). Notice that this is a categorical algebra in the sense of Definition 3.20 for the space $S = *$.

Definition 3.25 ([GH15, Definition 5.1.6])**.** Let S be a space. The *trivial categorical algebra in* $\mathcal{V}$ with objects S is given by the composite

$$\Delta^{\mathrm{op}}_S \to \Delta^{\mathrm{op}} \xrightarrow{I_{\mathcal{V}}} \mathcal{V}^{\otimes}.$$

We denote this categorical algebra by $E^{\mathcal{V}}_S$, or simply by E_S when $\mathcal{V}$ is clear from context.

Remark 3.26 ([GH15, Remark 5.1.7])**.** Consider the trivial categorical algebra E^n associated to the set $\{0, \dots, n\}$. This is the enriched ∞ category associated to the category with objects S and a unique morphism $i \to j$ for every $i, j \in \{0, \dots, n\}$. The categorical algebras E^n assemble into a cosimplicial categorical algebra $E^\bullet$. Let $\mathcal{C}$ be a categorical algebra in $\mathcal{V}$ with space of objects X. For $n = 0$, we have $E^0 = I_{\mathcal{V}}$, and a map $E^0 \to \mathcal{C}$ of categorical algebras is the same as a choice of point in X, by [GH15, Lemma 5.1.2]. For $n = 1$, we call a map of categorical algebras $E^1 \to \mathcal{C}$ an *equivalence* in $\mathcal{C}$ (c.f. [GH15, Definition 5.1.8], [GH15, Proposition 5.1.15]).

Definition 3.27. Let $\mathcal{V}^\otimes \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor between categorical algebras in $\mathcal{V}$, and let S and T be the spaces of objects of $\mathcal{C}$ and $\mathcal{D}$, respectively. The $\mathcal{V}$-functor F is:

- *fully faithful* if the maps $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ are equivalences in $\mathcal{V}$ for all $X, Y \in S$;
- *essentially surjective* if for every point $X \in T$ there exists an equivalence $E^1 \to \mathcal{D}$ from X to a point $E^0 \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ in the image of F.

3.5. **Categories enriched in monoidal ∞-categories.**

Theorem 3.28 ([GH15, Theorem 5.6.6])**.** *Let $\mathcal{V}^\otimes \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category. There is a full subcategory $\mathbf{Cat}_\infty^{\mathcal{V}}$ of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ such that the inclusion $\mathbf{Cat}_\infty^{\mathcal{V}} \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ has a left adjoint L exhibiting $\mathbf{Cat}_\infty^{\mathcal{V}}$ as a localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ with respect to the fully faithful and essentially surjective functors.*

Definition 3.29. A categorical algebra $\Delta_S^{\mathrm{op}} \to \mathcal{V}^\otimes$ that is in $\mathbf{Cat}_\infty^{\mathcal{V}}$ is a *complete* categorical algebra. We will call the complete categorical $\mathcal{V}$ algebras *$\mathcal{V}$-enriched ∞-categories.*

As noted in Remark 3.24, the term *$\mathcal{V}$-enriched ∞-categories* is sometimes used in [GH15] for categorical algebras in $\mathcal{V}$, Gepner and Haugseng sometimes use the term *complete $\mathcal{V}$-enriched ∞-categories* for objects of $\mathbf{Cat}_\infty^{\mathcal{V}}$. Our nomenclature agrees with more recent work on enriched ∞-categories, such as [Hei23].

3.6. **Change of enrichment.** The next lemma allows us to take a category enriched in $\mathcal{V}^\otimes$, and produce a category enriched in $\mathcal{W}^\otimes$ by means of a lax monoidal functor $\mathcal{V}^\otimes \to \mathcal{W}^\otimes$.

Lemma 3.30 ([GH15, Lemma 4.3.9])**.** *The construction $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is functorial in $\mathcal{V}$ with respect to lax monoidal functors of monoidal ∞-categories.*

Lemma 3.31 ([GH15, Lemma 5.7.5])**.** *If $\phi\colon \mathcal{V}^\otimes \to \mathcal{W}^\otimes$ is a lax monoidal functor, then the induced functor*

$$\phi_*\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})$$

preserves fully faithful and essentially surjective functors.

Proposition 3.32 ([GH15, Corollary 5.7.6])**.** *The construction $\mathbf{Cat}_\infty^{\mathcal{V}}$ is functorial in $\mathcal{V}$ with respect to lax monoidal functors of monoidal ∞-categories.*

Remark 3.33. Given a lax monoidal functor $\phi\colon \mathcal{V}^\otimes \to \mathcal{W}^\otimes$, the *change of enrichment* functor $\phi_*\colon \mathbf{Cat}_\infty^{\mathcal{V}} \to \mathbf{Cat}_\infty^{\mathcal{W}}$ of Proposition 3.32 is the composite

$$\mathbf{Cat}_\infty^{\mathcal{V}} \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \xrightarrow{\phi_*} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W}) \xrightarrow{L_{\mathcal{W}}} \mathbf{Cat}_\infty^{\mathcal{W}},$$

as noted in the proof of [GH15, Lemma 5.7.7].

Remark 3.34. Let $\mathcal{C} \in \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ be a categorical algebra in $\mathcal{V}^{\otimes}$ a monoidal ∞-category, and let $\phi\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ be a lax monoidal functor. By (the proof of) Theorem 3.28, the unit map $\mathcal{C} \to L\mathcal{C} \in \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is fully faithful and essentially surjective. By Lemma 3.31, $\phi_*\mathcal{C}$ is, up to equivalence in $\mathbf{Cat}_{\infty}^{\mathcal{W}}$, the same as localising $\mathcal{C}$ and then applying $\phi_*\colon \mathbf{Cat}_{\infty}^{\mathcal{V}} \to \mathbf{Cat}_{\infty}^{\mathcal{W}}$.

3.7. **Symmetric monoidal structures on enriched ∞-categories.**

Proposition 3.35 ([GH15, Corollary 5.7.12])**.** *Let $\mathcal{V}$ be a symmetric monoidal ∞-category. Then the symmetric monoidal structure on $\mathcal{V}$ makes $\mathbf{Cat}_{\infty}^{\mathcal{V}}$ into a symmetric monoidal ∞-category.*

Definition 3.36. Let $\mathbf{Pres}_{\infty}$ be the ∞-category of presentable ∞-categories and colimit preserving functors. There is a symmetric monoidal structure on $\mathbf{Pres}_{\infty}$, constructed in [Lur, Proposition 4.8.1.15]. Maps $A \otimes B \to C$ out of the tensor product of two presentable ∞-categories correspond to functors $A \times B \to C$ that preserve colimits separately in each component. A *presentably monoidal ∞-category* is an associative algebra object in $\mathbf{Pres}_{\infty}^{\otimes}$.

Remark 3.37. The inclusion $\mathbf{Pres}_{\infty} \to \mathbf{Cat}_{\infty}$ induces a lax symmetric monoidal functor $\mathbf{Pres}_{\infty}^{\otimes} \to \mathbf{Cat}_{\infty}^{\times}$, by [Lur, Corollary 4.8.1.4].

Proposition 3.38 ([GH15, Corollary 5.4.5])**.** *Let $\mathcal{V}$ be a presentably monoidal ∞-category. Then $\mathbf{Cat}_{\infty}^{\mathcal{V}}$ is presentable.*

Corollary 3.39. *Let $\mathcal{V}$ be a cartesian symmetric monoidal ∞-category that is presentably monoidal. The symmetric monoidal structure on $\mathbf{Cat}_{\infty}^{\mathcal{V}}$ induced via Proposition 3.35 is Cartesian.*

Proof. In the text immediately after [GH15, Proposition 3.6.14], Gepner and Haugseng construct an *exterior product*

$$\boxtimes\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{A}) \times \mathrm{Alg}_{\mathrm{cat}}(\mathcal{B}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{A} \times \mathcal{B})$$

for any two monoidal ∞-categories $\mathcal{A}$ and $\mathcal{B}$. Denote now the product functor in $\mathcal{V}$ by $\mu\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, which is right adjoint to the diagonal $\Delta\colon \mathcal{V} \to \mathcal{V} \times \mathcal{V}$. By [GH15, Lemma 4.3.19], we obtain an adjunction

$$\Delta_*\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \rightleftarrows \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V} \times \mathcal{V})\colon \mu_*.$$

As noted in the proof of [GH15, Proposition 5.7.14], the tensor product of $\mathcal{C}, \mathcal{D} \in \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is given by $\mu_*(\mathcal{C} \boxtimes \mathcal{D})$. Let $\mathcal{E} \in \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. By adjunction, we have an equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\mathcal{E}, \mu_*(\mathcal{C} \boxtimes \mathcal{D})) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V} \times \mathcal{V})}(\Delta_*\mathcal{E}, \mathcal{C} \boxtimes \mathcal{D}).$$

Consider the functors

$$\pi_{1,*}, \pi_{2,*}\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V} \times \mathcal{V}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$$

induced by the projections $\pi_1, \pi_2\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. By the proof of [GH15, Lemma 5.7.10], we have a chain of equivalences

$$\begin{aligned}
&\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V} \times \mathcal{V})}(\Delta_*\mathcal{E}, \mathcal{C} \boxtimes \mathcal{D}) \\
&\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\pi_{1,*}\Delta_*\mathcal{E}, \mathcal{C}) \times \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\pi_{1,*}\Delta_*\mathcal{E}, \mathcal{D}) \\
&\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\mathcal{E}, \mathcal{C}) \times \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\mathcal{E}, \mathcal{D})
\end{aligned}$$

showing that the tensor product in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is the Cartesian product of categorical algebras. The inclusion $\mathbf{Cat}_\infty^{\mathcal{V}}$ is right adjoint, so that if $\mathcal{C}$ and $\mathcal{D}$ are in $\mathbf{Cat}_\infty^{\mathcal{V}}$, their product in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ computes the product in $\mathbf{Cat}_\infty^{\mathcal{V}}$. The tensor product of the $\mathcal{V}$-enriched ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is the localization of the tensor product in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$, which is thus also the product. $\square$

Proposition 3.40. *Let $\phi\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ be a lax symmetric monoidal functor. The induced functor $\phi_*\colon \mathbf{Cat}_\infty^{\mathcal{V},\otimes} \to \mathbf{Cat}_\infty^{\mathcal{W},\otimes}$ is lax symmetric monoidal.*

Proof. This is a special case of [Law23, Theorem 1.1]. $\square$

3.8. **Graphs of categorical algebras.**

Definition 3.41 ([GH15, Definition 4.3.3])**.** Let $\mathcal{V}$ be a monoidal ∞-category. Let $\mathcal{F}_{\mathcal{V}} \to \boldsymbol{\mathcal{S}}$ be the Cartesian fibration associated to the functor $\boldsymbol{\mathcal{S}} \to \mathbf{Cat}_\infty$ defined by the assignment $S \mapsto \mathrm{Fun}(S, \mathcal{V})$. The ∞-category of $\mathcal{V}$-*graphs* is the total space of the Cartesian fibration defined by the pullback

$$\begin{tikzcd} \mathrm{Graph}_\infty(\mathcal{V}) \arrow[r] \arrow[d] & \mathcal{F}_{\mathcal{V}} \arrow[d] \\ \boldsymbol{\mathcal{S}} \arrow[r, "\Delta"'] & \boldsymbol{\mathcal{S}}, \end{tikzcd}$$

where $\Delta\colon \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{S}}$ is the functor defined by the assignment $S \mapsto S \times S$.

Notice that the fiber of the Cartesian fibration $\mathrm{Graph}_\infty(\mathcal{V}) \to \boldsymbol{\mathcal{S}}$ above a space S is $\mathrm{Fun}(S \times S, \mathcal{V})$. One can forget some of the structure of a categorical algebra to end up with a graph on the underlying space.

Remark 3.42 ([GH15, Remark 4.3.4])**.** There is a forgetful functor $\tau^*\colon \mathrm{Alg}_{\Delta_S^{\mathrm{op}}}(\mathcal{V}) \to \mathrm{Fun}(S \times S, \mathcal{V})$ and a functor $\tau_!\colon \mathrm{Fun}(S \times S, \mathcal{V}) \to \mathrm{Alg}_{\Delta_S^{\mathrm{op}}}(\mathcal{V})$ which assemble into an adjunction

$$\mathcal{F}\colon \mathrm{Graph}_\infty(\mathcal{V}) \leftrightarrows \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})\colon \mathcal{U}.$$

Lemma 3.43 ([GH15, Lemma A.5.5])**.** *The forgetful functor $\tau^*\colon \mathrm{Alg}_{\Delta_S^{\mathrm{op}}}(\mathcal{V}) \to \mathrm{Fun}(S \times S, \mathcal{V})$ is conservative.*

Our main use of the category of $\mathcal{V}$-graphs will be in pullback computations. These are easy to compute in the fibers $\mathrm{Fun}(S \times S, \mathcal{V})$, and it turns out we can compute any pullback by replacing it by one inside a fiber.

Proposition 3.44. *Let $F\colon A \to B$ be a Cartesian fibration of ∞-categories. Suppose B has pullbacks, that each fiber A_b has pullbacks, and that the Cartesian transport functors preserve pullbacks in the fibers. Then the ∞-category A has pullbacks, and F preserves them.*

Proof. Let $x \to y \leftarrow z$ be a span in A. Compute the pullback ξ of

$$F(x) \to F(y) \leftarrow F(z)$$

in B. Choose cartesian lifts of the maps $\xi \to F(x)$, $\xi \to F(y)$ and $\xi \to F(z)$, yielding a solid diagram

$$\begin{tikzcd}
p \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & x_\xi \arrow[d] \arrow[dr] & \\
z_\xi \arrow[r] \arrow[dr] & y_\xi \arrow[dr] & x \arrow[d] \\
& z \arrow[r] & y.
\end{tikzcd}$$

Take the pullback p in A_ξ, denoted in the diagram with the dashed arrows. By the dual of [Lur08, Proposition 4.3.1.10], the fact that the Cartesian transport functors preserve pullbacks implies that p is an F-pullback. But $F(p)$ is a pullback, so p is a pullback by [Lur08, Proposition 4.3.1.5(2)]. Now p is also a pullback of the original span $x \to y \leftarrow z$, by immediate application of [Lur08, Proposition 4.3.1.9].

□

Remark 3.45. The proof of Proposition 3.44 gives a recipe for computing pullbacks in A.

(1) Given a span $x \to y \leftarrow z$ in A, compute the pullback ξ of
$$F(x) \to F(y) \leftarrow F(z)$$
in B.

(2) Use Cartesian lifts to produce a span
$$x_\xi \to y_\xi \leftarrow z_\xi$$
in A_ξ and compute its A_ξ-pullback p. This is also the pullback of the original span in A.

Corollary 3.46. *Let $\mathcal{V}$ be an ∞-category with pullbacks. The ∞-category $\mathrm{Graph}_\infty(\mathcal{V})$ has pullbacks and the Cartesian fibration $\mathrm{Graph}_\infty(\mathcal{V}) \to \boldsymbol{S}$ preserves them. Moreover, pullback diagrams in $\mathrm{Graph}_\infty(\mathcal{V})$ can be computed by pulling them back to a fiber.*

Proof. Since $\mathcal{V}$ has pullbacks, so do the functor categories $\mathrm{Fun}(S \times S, \mathcal{V})$, and the Cartesian transport functors are given by composition. □

4. Left modules over associative algebras

In this section we recall how some of the theory of left modules over associative algebras developed in [Lur, Chapter 4.2] is used by [Hei23] to provide an alternative model of enriched ∞-categories. We will in particular be interested in computing products in the various categories of left modules that [Hei23] makes use of. This will be of later importance in Section 6 when we construct commutative algebra objects of $\mathbf{Cat}_\infty^{\mathcal{V}}$ with its Cartesian symmetric monoidal structure. We closely follow [Hei23].

4.1. Left tensored ∞-categories.

Definition 4.1 ([Hei23, 1.2.4])**.** Let S be an ∞-category and $\mathcal{E} \subset \mathrm{Fun}([1], S)$ a full subcategory. A functor $\phi\colon \mathcal{C} \to S$ is a *coCartesian fibration relative to $\mathcal{E}$* if the pullback $\phi'\colon [1] \times_S \mathcal{C}$ along any functor $[1] \to S$ corresponding to a morphism of $\mathcal{E}$ is a coCartesian fibration and the projection $[1] \times_S \mathcal{C} \to \mathcal{C}$ sends ϕ'-coCartesian morphisms to ϕ-coCartesian morphisms.

Definition 4.2 ([Hei23, Notation 11.1])**.** Let S be an ∞-category and $\mathcal{E} \subset \mathrm{Fun}([1], S)$ a full subcategory. Let $\mathbf{Cat}^{\mathcal{E}}_{\infty/S} \subset \mathbf{Cat}_{\infty/S}$ be the subcategory spanned by the coCartesian fibrations relative to $\mathcal{E}$ and functors over S preserving coCartesian lifts of morphisms in $\mathcal{E}$.

Definition 4.3 ([Hei23, Notation 3.1])**.** A morphism in Δ^{op} *preserves the maximum* if it corresponds to a map $[m] \to [n]$ in Δ sending m to n.

Definition 4.4 ([Hei23, Notation 3.25, Notation 3.27])**.**

- Denote by $\mathbf{Cat}^{\max}_{\infty/\Delta^{\mathrm{op}}}$ the subcategory of $\mathbf{Cat}_{\infty/\Delta^{\mathrm{op}}}$ spanned by the coCartesian fibrations relative to the inert morphisms preserving the maximum and functors over Δ^{op} preserving their coCartesian lifts, as in Definition 4.2.
- Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a coCartesian fibration relative to the inert morphisms preserving the maximum. We denote by $\mathbf{Cat}^{\max}_{\infty/\mathcal{V}^{\otimes}}$ the subcategory of $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}$ of coCartesian fibrations relative to the collection of coCartesian lifts in $\mathcal{V}^{\otimes}$ of inert morphisms of Δ^{op} that preserve the maximum.
- Denote by $\mathbf{Cat}^{\mathrm{cocart}}_{\infty/\Delta^{\mathrm{op}}}$ the subcategory $\mathbf{Cat}_{\infty/\Delta^{\mathrm{op}}}$ spanned by the coCartesian fibrations and functors preserving coCartesian lifts of maps in Δ^{op}.
- Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a generalised monoidal ∞-category. Define $\mathbf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{V}^{\otimes}}$ to be the subcategory of $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}$ spanned by the coCartesian fibrations relative to the coCartesian lifts in $\mathcal{V}^{\otimes}$.

Definition 4.5 ([Hei23, Definition 3.11])**.** Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a generalised nonsymmetric ∞-operad. Let $\phi\colon \mathcal{M}^{\circledast} \to \mathcal{V}^{\otimes}$ be a map of coCartesian fibrations relative to the collection of inert morphisms of Δ^{op} that preserve the maximum. We call ϕ an *∞-category weakly tensored over $\mathcal{V}$* if the following conditions hold:

(1) For every $n \geq 0$ the map $[0] \simeq \{n\} \subset [n]$ induces an equivalence
$$\mathcal{M}^{\circledast}_{[n]} \to \mathcal{V}^{\otimes}_{[n]} \times_{\mathcal{V}^{\otimes}_{[0]}} \mathcal{M}^{\circledast}_{[0]}.$$

(2) For every $X, Y \in \mathcal{M}^{\circledast}$ lying over $[m], [n] \in \Delta^{\mathrm{op}}$, the coCartesian lift $Y \to Y'$ of the map $[0] \simeq \{n\} \subset [n]$ in Δ induces a pullback square
$$\begin{tikzcd}
\mathcal{M}^{\circledast}(X, Y) \arrow[r] \arrow[d] & \mathcal{M}^{\circledast}(X, Y') \arrow[d] \\
\mathcal{V}^{\otimes}(\phi(X), \phi(Y)) \arrow[r] & \mathcal{V}^{\otimes}(\phi(X), \phi(Y')).
\end{tikzcd}$$

Note that $\phi\colon \mathcal{M}^{\circledast} \to \mathcal{V}^{\otimes}$ is an object of $\mathbf{Cat}^{\max}_{\infty/\mathcal{V}^{\otimes}}$. Indeed, we have an equivalence $\mathbf{Cat}^{\max}_{\infty/\mathcal{V}^{\otimes}} \simeq (\mathbf{Cat}^{\max}_{\infty/\Delta^{\mathrm{op}}})_{/\mathcal{V}^{\otimes}}$, see the proof of [Hei23, Proposition 3.32]. We denote by $\omega\,\mathrm{LMod}_{\mathcal{V}}$ the full subcategory of $\mathbf{Cat}^{\max}_{\infty/\mathcal{V}^{\otimes}}$ spanned by the ∞-categories weakly tensored over $\mathcal{V}$.

Notation 4.6. Let $\mathcal{M}^{\circledast} \to \mathcal{V}$ be an ∞-category weakly tensored over a generalised nonsymmetric ∞-operad $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$. We will write $\mathcal{M} := \mathcal{M}_{[0]}$ and think of it as the ∞-category of *objects* of $\mathcal{M}^{\circledast}$. We view $\mathcal{V}$ as *acting* on $\mathcal{M}$.

Definition 4.7. Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category. An ∞-category $\phi\colon \mathcal{M}^{\circledast} \to \mathcal{V}^{\otimes}$ weakly left tensored over $\mathcal{V}$ *exhibits $\mathcal{M}$ as left tensored over $\mathcal{V}$* if ϕ is a map of coCartesian fibrations over Δ^{op}, i.e. if ϕ is a map in $\mathbf{Cat}^{\mathrm{cocart}}_{\infty/\Delta^{\mathrm{op}}}$. Again by the proof of [Hei23, Proposition 3.32], we have an equivalence $\mathbf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{V}^{\otimes}} \simeq$

$(\mathbf{Cat}_{\infty/\Delta^{\mathrm{op}}}^{\mathrm{cocart}})_{/\mathcal{V}^{\otimes}}$, so we define $\mathrm{LMod}_{\mathcal{V}}$ as the full subcategory of $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}^{\mathrm{cocart}}$ spanned by the ∞-categories left tensored over $\mathcal{V}$.

Remark 4.8 ([Hei23, Remark 3.30])**.** Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category. There is an equivalence $\mathrm{LMod}_{\mathcal{V}} \simeq \mathrm{LMod}_{\mathcal{V}}(\mathbf{Cat}_{\infty})$ between the ∞-category of Definition 4.7 and the ∞-category of left $\mathcal{V}$-modules defined by Lurie in [Lur, Remark 4.2.1.21].

Proposition 4.9. *Let $f\colon A \to S$ and $g\colon B \to S$ be coCartesian fibrations relative to a full subcategory $\mathcal{E} \subset \mathrm{Fun}([1], S)$. Then their $\mathbf{Cat}_{\infty}$-fiber product in $A \times_S B \to S$ is a coCartesian fibration relative to $\mathcal{E}$.*

Proof. The map $(f, g)\colon A \times B \to S \times S$ is a coCartesian fibration relative to $\mathcal{E} \times \mathcal{E}$. Let $[1] \to S$ be an arrow in $\mathcal{E}$. Consider the pullback squares

$$\begin{tikzcd}
(A \times_S B) \times_S [1] \arrow[r] \arrow[d, "\phi'"] & A \times_S B \arrow[r] \arrow[d, "\phi"] & A \times B \arrow[d] \\
{[1]} \arrow[r] & S \arrow[r, "\Delta"'] & S \times S,
\end{tikzcd}$$

where the bottom right map is the diagonal. Using the outer rectangle, we see that the map $\phi'\colon (A \times_S B) \times_S [1] \to [1]$ is a coCartesian fibration. We need to check that the map $(A \times_S B) \times_S [1] \to A \times_S B$ sends ϕ'-coCartesian arrows to ϕ-coCartesian arrows. The composite $(A \times_S B) \times_S [1] \to A \times B$ sends ϕ'-coCartesian arrows to (f, g)-coCartesian arrows. We conclude by noting that arrows in $A \times_S B$ mapping to (f, g)-coCartesian arrows are ϕ-coCartesian arrows. ☐

Proposition 4.10. *Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a generalised nonsymmetric ∞-operad. The ∞-category $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}^{\max}$ has products computed in $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}$. Similarly, the full subcategory $\omega\,\mathrm{LMod}_{\mathcal{V}} \subset \mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}^{\max}$ has products computed in $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}$.*

Proof. The first part follows by Proposition 4.9, the second is an easy check of the defining properties of a weakly tensored ∞-category. ☐

Proposition 4.11. *Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category. The ∞-category $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}^{\mathrm{cocart}}$ has products computed in $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}$. Similarly, the full subcategory $\mathrm{LMod}_{\mathcal{V}} \subset \mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}^{\mathrm{cocart}}$ has products computed in $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}$.*

4.2. **Modules as models of enriched ∞-categories.**

Definition 4.12 ([Hei23, Notation 3.18])**.** Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be an ∞-operad. Let $(V_1, \ldots, V_n) \in \mathcal{V}^{\times n} \simeq \mathcal{V}_{[n]}$ and let $W \in \mathcal{V}$, for some $n \geq 0$. The space of *multi-morphisms* from $(V_1, \ldots, V_n)$ to W in $\mathcal{V}$, denoted

$$\mathrm{Mul}_{\mathcal{V}}(V_1, \ldots, V_n; W),$$

is the subspace of $\mathcal{V}^{\otimes}((V_1, \ldots, V_n), W)$ spanned by the maps over the active morphism $[1] \to [n] \in \Delta$.

Let $\mathcal{M}^{\circledast} \to \mathcal{V}^{\otimes}$ be an ∞-category weakly left tensored over $\mathcal{V}$, let $(V_1, \ldots, V_n) \in \mathcal{V}^{\times n}$ and let $X, Y \in \mathcal{M}$. The space of multi-morphisms from $(V_1, \ldots, V_n, X)$ to Y in $\mathcal{M}$, denoted

$$\mathrm{Mul}_{\mathcal{M}}(V_1, \ldots, V_n, X; Y),$$

is the subspace of $\mathcal{M}^{\otimes}((V_1, \ldots, V_n, X), Y)$ spanned by the maps over the morphism $[0] \to [n] \in \Delta$ defined by the assignment $0 \mapsto 0$.

Definition 4.13 ([GH15, Definition 3.121])**.** Let $\mathcal{M}^{\circledast} \to \mathcal{V}^{\otimes}$ be an ∞-category weakly left tensored over a nonsymmetric ∞-operad. A *morphism object* of X, Y in $\mathcal{M}$ is an object $\mathrm{Mor}_{\mathcal{M}}(X, Y) \in \mathcal{V}$ together with a multimorphism

$$\alpha \in \mathrm{Mul}_{\mathcal{M}}(\mathrm{Mor}_{\mathcal{M}}(X, Y), X; Y)$$

such that for any collection of objects $Z_1, \dots, Z_n$ in $\mathcal{V}$, the map

$$\mathrm{Mul}_{\mathcal{V}}(Z_1, \dots, Z_n; \mathrm{Mor}_{\mathcal{M}}(X, Y)) \to \mathrm{Mul}_{\mathcal{M}}(Z_1, \dots, Z_n, X; Y)$$

induced by α is an equivalence.

Remark 4.14. The map

$$\mathrm{Mul}_{\mathcal{V}}(Z_1, \dots, Z_n; \mathrm{Mor}_{\mathcal{M}}(X, Y)) \to \mathrm{Mul}_{\mathcal{M}}(Z_1, \dots, Z_n, X; Y)$$

in Definition 4.13 is induced from a composition of maps on mapping spaces, as follows. Firstly, we obtain a map

$$\mathcal{V}^{\otimes}((Z_1, \dots, Z_n), \mathrm{Mor}_{\mathcal{M}}(X, Y)) \to \mathcal{M}^{\circledast}((Z_1, \dots, Z_n, X), (\mathrm{Mor}_{\mathcal{M}}(X, Y), X))$$

from the pullback square

$$\begin{tikzcd}
\mathcal{M}^{\circledast}((Z_1, \dots, Z_n, X), (\mathrm{Mor}_{\mathcal{M}}(X, Y), X)) \arrow[r] \arrow[d] & \mathcal{M}^{\circledast}((Z_1, \dots, Z_n, X), X) \arrow[d] \\
\mathcal{V}^{\otimes}((Z_1, \dots, Z_n), \mathrm{Mor}_{\mathcal{M}}(X, Y)) \arrow[r] & V^{\otimes}((Z_1, \dots, Z_n), *)
\end{tikzcd}$$

of Definition 4.5, the identity map, and the constant map to the coCartesian projection $(Z_1, \dots, Z_n, X) \to X$. Secondly, the map

$$\mathcal{M}^{\circledast}((Z_1, \dots, Z_n, X), (\mathrm{Mor}_{\mathcal{M}}(X, Y), X)) \to \mathcal{M}^{\circledast}((Z_1, \dots, Z_n, X), Y)$$

is induced via composition with $\alpha \in \mathcal{M}^{\circledast}((\mathrm{Mor}_{\mathcal{M}}(X, Y), X), Y)$.

Definition 4.15 ([GH15, Definition 3.122])**.** Let $\mathcal{M}^{\circledast} \to \mathcal{V}^{\otimes}$ be an ∞-category weakly tensored over a nonsymmetric ∞-operad. If for every pair of objects $X, Y \in \mathcal{M}$ there is a morphism object, we say $\mathcal{M}$ is *enriched in* $\mathcal{V}$.

Remark 4.16. In the case of ∞-categories weakly tensored over a monoidal ∞-category, Definition 4.15 agrees with Lurie's [Lur, Definition 4.2.1.28], by [Hei23, Corollary 3.124].

Definition 4.17. Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category. Let $\mathbf{Cat}_{\infty}^{\mathcal{V},\mathrm{Lur}}$ be the full subcategory of $\omega\,\mathrm{LMod}_{\mathcal{V}}$ spanned by the ∞-categories enriched in $\mathcal{V}$.

Proposition 4.18. *Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category such that $\mathcal{V}$ has finite products. Then $\mathbf{Cat}_{\infty}^{\mathcal{V},\mathrm{Lur}}$ has finite products that can be computed in $\omega\,\mathrm{LMod}_{\mathcal{V}}$.*

Proof. Let $\mathcal{M}, \mathcal{N}$ be enriched in $\mathcal{V}$. Write $\mathcal{P}^{\circledast} \to \mathcal{V}^{\otimes}$ for their product in $\omega\,\mathrm{LMod}_{\mathcal{V}}$. We have $\mathcal{P}_{[0]} \simeq \mathcal{M}_{[0]} \times \mathcal{N}_{[0]}$. Let $X := (X_{\mathcal{M}}, X_{\mathcal{N}})$ and $Y := (Y_{\mathcal{M}}, Y_{\mathcal{N}})$ be in $\mathcal{P}_{[0]}$. We define the morphism object

$$\mathrm{Mor}_{\mathcal{P}}(X, Y) := \mathrm{Mor}_{\mathcal{M}}(X_{\mathcal{M}}, Y_{\mathcal{M}}) \times \mathrm{Mor}_{\mathcal{N}}(X_{\mathcal{N}}, Y_{\mathcal{N}}) \quad \in \mathcal{V}.$$

Let $\alpha_{\mathcal{M}}$ and $\alpha_{\mathcal{N}}$ be the maps implementing the equivalences for the morphism objects $\mathrm{Mor}_{\mathcal{M}}(X_{\mathcal{M}}, Y_{\mathcal{M}})$ and $\mathrm{Mor}_{\mathcal{N}}(X_{\mathcal{N}}, Y_{\mathcal{N}})$ in $\mathcal{V}$ respectively. Via the projections

$\pi_{\mathcal{M}}\colon \mathrm{Mor}_{\mathcal{P}}(X,Y) \to \mathrm{Mor}_{\mathcal{M}}(X_{\mathcal{M}}, Y_{\mathcal{M}})$ and $\pi_{\mathcal{N}}\colon \mathrm{Mor}_{\mathcal{P}}(X,Y) \to \mathrm{Mor}_{\mathcal{N}}(X_{\mathcal{N}}, Y_{\mathcal{N}})$ we obtain a map

$$\begin{aligned}
&\alpha \in P^{\circledast}((\mathrm{Mor}_{\mathcal{P}}(X,Y), X), Y) \\
&\quad \simeq \mathcal{M}^{\circledast}((\mathrm{Mor}_{\mathcal{P}}(X,Y), X_{\mathcal{M}}), Y_{\mathcal{M}}) \times_{\mathcal{V}^{\otimes}(\mathrm{Mor}_{\mathcal{P}}(X,Y),*)} \mathcal{N}^{\circledast}((\mathrm{Mor}_{\mathcal{P}}(X,Y), X_{\mathcal{N}}, Y_{\mathcal{N}}))
\end{aligned}$$

implementing the needed equivalence. □

Definition 4.19. Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category. A left module

$$\mathcal{M} \in \mathrm{LMod}_{\mathcal{V}}(\mathbf{Cat}_{\infty})$$

is said to have *closed left $\mathcal{V}$-action* if for any object X of $\mathcal{M}$, the tensor product functor $(-) \otimes X \colon \mathcal{V} \to \mathcal{M}$ has a right adjoint, denoted by $\mathrm{Map}(X, -)\colon \mathcal{M} \to \mathcal{V}$.

Remark 4.20. By [Hei23, Example 3.128], a left tensored ∞-category $\mathcal{M}^{\circledast} \to \mathcal{V}^{\otimes}$ exhibits $\mathcal{M}$ as $\mathcal{V}$-enriched if and only if the left action is closed.

Remark 4.21. By [Lur, Corollary 4.2.3.3], a small diagram in $\mathrm{LMod}_{\mathcal{V}}(\mathbf{Cat}_{\infty})$ is a limit if and only if it is a limit in $\mathbf{Cat}_{\infty}$ when composed with the forgetful functor $\mathrm{LMod}_{\mathcal{V}}(\mathbf{Cat}_{\infty}) \to \mathbf{Cat}_{\infty}$ that sends a left $\mathcal{V}$-module $\mathcal{M}$ to its underlying ∞-category.

4.3. **An equivalence between models of enriched ∞-categories.**

Definition 4.22. A monoidal ∞-category $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ is said to be *compatible with small colimits* if $\mathcal{V}$ admits small colimits and the tensor product $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ preserves small colimits in each variable.

Theorem 4.23 ([Hei23, Theorem 1.9])**.** *Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category compatible with small colimits. There is an equivalence*

$$\mathbf{Cat}_{\infty}^{\mathcal{V},\mathrm{Lur}} \simeq \mathbf{Cat}_{\infty}^{\mathcal{V}}.$$

Corollary 4.24. *Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category compatible with small colimits. Let $A \in \mathrm{CAlg}(\mathbf{Cat}_{\infty}^{\mathcal{V},\mathrm{Lur}})$ be a commutative algebra object of $\mathbf{Cat}_{\infty}^{\mathcal{V},\mathrm{Lur}}$ viewed as a Cartesian symmetric monoidal ∞-category. Viewing also $\mathbf{Cat}_{\infty}^{\mathcal{V}}$ as a Cartesian symmetric monoidal ∞-category, the equivalence of Theorem 4.23 yields a commutative algebra object of $\mathbf{Cat}_{\infty}^{\mathcal{V}}$.*

Remark 4.25. Let $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal ∞-category and suppose $\mathcal{V}$ has finite products. In the previous section we showed that products in $\mathrm{LMod}_{\mathcal{V}}$ and in $\mathbf{Cat}_{\infty}^{\mathcal{V},\mathrm{Lur}}$ are computed in $\mathbf{Cat}_{\infty/\mathcal{V}^{\otimes}}$. Let $A\colon \mathbf{Fin}_{*} \to \mathrm{LMod}_{\mathcal{V}}$ be a commutative algebra object of $\mathrm{LMod}_{\mathcal{V}}$ with its Cartesian symmetric monoidal structure, and suppose that $A(\langle 1\rangle)$ has closed left action of $\mathcal{V}$. Then A can be viewed as a commutative algebra object of $\mathbf{Cat}_{\infty}^{\mathcal{V},\mathrm{Lur}}$.

Remark 4.26. As a special case of Theorem 4.23, consider an ∞-topos $\mathcal{V}$. Then, viewed as a monoidal ∞-category via the Cartesian product, $\mathcal{V}$ is a left module over itself with closed action. The right adjoints are given by internal homs. The space of objects of the corresponding $\mathcal{V}$-enriched ∞-category is the core groupoid of $\mathcal{V}$, and the mapping objects are the internal hom objects of $\mathcal{V}$, c.f. [GH15, Corollary 7.4.10].

5. Orbifolds enriched in open embeddings

In Section 2, we constructed ∞-categories $\mathbf{Orb}_n$ and $\mathbf{ODisk}_n$ of n-orbifolds and of n-orbifold disks, respectively. These are full subcategories of $\mathrm{Shv}(\mathbf{Man})$. We would now like to construct new ∞-categories of orbifolds where the morphisms are representable open embeddings, and the higher morphisms are sort of *isotopies*: homotopies of open embeddings through open embeddings.

In Section 5.1 we will construct *embedding stacks*, substacks of the mapping stacks which encode the representable open embeddings between two orbifolds and the (higher) isotopies between them. In Section 5.2 we will use these to construct an *enriched category of orbifolds* $\mathbf{Orb}_n^E$, with orbifolds as objects and embedding stacks as mapping objects. In Section 5.3 we will use the *shape* functor

$$\Pi_\infty \colon \mathrm{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}},$$

which we can think of as sending a stack to its *homotopy type*, to change the enrichment of $\mathbf{Orb}_n^E$ to an enrichment in spaces. Since an ∞-category enriched in $\boldsymbol{\mathcal{S}}$ is just an ∞-category ([GH15, Theorem 5.4.6]), this yields an honest ∞-category of orbifolds.

5.1. **Fiberwise open substacks.**

Definition 5.1. Let $\mathcal{X}, \mathcal{Y} \in \mathrm{Shv}(\mathbf{Man})$ be stacks. For a given manifold T, a map $f \colon \mathcal{X} \times T \to \mathcal{Y}$ is called a *T-fiberwise open substack* if the map

$$\mathcal{X} \times T \xrightarrow{(f,\mathrm{pr}_T)} \mathcal{Y} \times T$$

exhibits $\mathcal{X} \times T$ as an open substack of $\mathcal{Y} \times T$.

Remark 5.2. Notice that if $\mathcal{X} \times T \rightrightarrows \mathcal{Y}$ are equivalent maps in $\mathrm{Map}(\mathcal{X}, \mathcal{Y})(T) \in \boldsymbol{\mathcal{S}}$, then one map is a T-fiberwise open substack if and only if the other is.

Definition 5.3. Let $\mathcal{X}, \mathcal{Y} \in \mathrm{Shv}(\mathbf{Man})$ be stacks. For each $T \in \mathbf{Man}$, let

$$\mathrm{Emb}(\mathcal{X}, \mathcal{Y})(T) \hookrightarrow \mathrm{Map}(\mathcal{X}, \mathcal{Y})(T)$$

be the union of connected components of $\mathrm{Map}(\mathcal{X}, \mathcal{Y})(T)$ spanned by the T-fiberwise open substacks.

Proposition 5.4. *The assignment $T \mapsto \mathrm{Emb}(\mathcal{X}, \mathcal{Y})(T)$ defines a substack $\mathrm{Emb}(\mathcal{X}, \mathcal{Y})$ of the mapping stack $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$.*

Proof. We need to show this assignment is a presheaf, and that it satisfies descent with respect to open covers. Let $g \colon T' \to T$ be a map of manifolds. For any $f : \mathcal{X} \times T \to \mathcal{Y}$, the square

$$\begin{array}{ccc} \mathcal{X} \times T' & \xrightarrow{(h,\mathrm{pr}_{T'})} & \mathcal{Y} \times T' \\ \downarrow & & \downarrow \\ \mathcal{X} \times T & \xrightarrow[(f,\mathrm{pr}_T)]{} & \mathcal{Y} \times T \end{array}$$

is a pullback, where

$$h : \mathcal{X} \times T' \to \mathcal{X} \times T \to \mathcal{Y}$$

is a composition of $(\mathrm{id}_{\mathcal{X}}, g)$ and f. Since open substacks are closed under pullbacks by Remark 2.13, we see that h is a T'-fiberwise open substack.

To prove that $\mathrm{Emb}(\mathcal{X}, \mathcal{Y})$ is a sheaf, fix $T \in \mathbf{Man}$ and let $\mathcal{U}$ be an open cover of T. We will show the top horizontal map in the diagram

$$\begin{tikzcd} \mathrm{Emb}(\mathcal{X}, \mathcal{Y})(T) \arrow[r] \arrow[d, hook] & \lim(\mathrm{Emb}(\mathcal{X}, \mathcal{Y})(\mathcal{U}_\bullet)) \arrow[d, hook] \\ \mathrm{Map}(\mathcal{X}, \mathcal{Y})(T) \arrow[r, "\simeq"] & \lim(\mathrm{Map}(\mathcal{X}, \mathcal{Y})(\mathcal{U}_\bullet)) \end{tikzcd}$$

is an equivalence. The bottom horizontal map is an equivalence because $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ is a stack, and the downward maps are monomorphisms by definition of $\mathrm{Emb}(\mathcal{X}, \mathcal{Y})$. It thus suffices to show that the induced map $\pi_0 \mathrm{Emb}(\mathcal{X}, \mathcal{Y})(T) \to \pi_0 \lim(\mathrm{Emb}(\mathcal{X}, \mathcal{Y})(\mathcal{U}_\bullet))$ is a surjection of sets. Given $[f] \in \pi_0 \lim(\mathrm{Emb}(\mathcal{X}, \mathcal{Y}))(\mathcal{U}_\bullet)$, we need to show that the corresponding class $[f] \in \pi_0 \mathrm{Map}(\mathcal{X}, \mathcal{Y})(T)$ is actually in $\pi_0 \mathrm{Emb}(\mathcal{X}, \mathcal{Y})(T)$. To this end, choose a representative $f \colon \mathcal{X} \times T \to \mathcal{Y}$. We need to show the map $(f, \mathrm{pr}_T) \colon \mathcal{X} \times T \to \mathcal{Y} \times T$ is an open substack. Note that for each $U \in \mathcal{U}$, the restriction $\mathcal{X} \times U \to \mathcal{Y}$ is in $\mathrm{Emb}(\mathcal{X}, \mathcal{Y})(U)$ because the diagram

$$\begin{tikzcd} & \lim(\mathrm{Emb}(\mathcal{X}, \mathcal{Y})(\mathcal{U}_\bullet)) \arrow[r] \arrow[d] & \mathrm{Emb}(\mathcal{X}, \mathcal{Y})(U) \arrow[d, hook] \\ \mathrm{Map}(\mathcal{X}, \mathcal{Y})(T) \arrow[r, "\simeq"] & \lim(\mathrm{Map}(\mathcal{X}, \mathcal{Y})(\mathcal{U}_\bullet)) \arrow[r] & \mathrm{Map}(\mathcal{X}, \mathcal{Y})(U) \end{tikzcd}$$

commutes. Let $M \to \mathcal{Y} \times T$ be a map from a manifold M. We observe that pulling back M along the open substack $\mathcal{Y} \times U \to \mathcal{Y} \times T$ gives an open cover of M. The result now follows from the fact that open embeddings of manifolds can be checked on open covers of the target.

□

Proposition 5.5. *Let S be the core groupoid of* $\mathrm{Shv}(\mathbf{Man})$*. The embedding substack construction defines a subfunctor*

$$\mathrm{Emb}(-,-) \colon S^{\mathrm{op}} \times S \to \mathrm{Shv}(\mathbf{Man})$$

of $\mathrm{Map}(-,-)$.

5.2. **Enriching stacks in fiberwise open substacks.** The goal of this section is to construct a $\mathrm{Shv}(\mathbf{Man})$-enriched ∞-category of orbifolds, with mapping objects the embedding stacks. We begin with two technical propositions that will be crucial in the construction.

Proposition 5.6. *Let C be an ∞-category and let $\sigma \colon \Lambda^m_m \to C$ be a right horn, for $m \geq 3$. Suppose that*

$$f \colon \Delta^1 \xrightarrow{(m-1,m)} \Lambda^m_m \to C$$

is a monomorphism. Then σ admits a filler $\Delta^m \to C$, and the space of such fillers is contractible.

Proof. That a filler exists is one direction of [Lur18, Tag 05FY]. It remains to show contractibility of the space of fillers. Writing $f \colon x \to y \in C$, by the equivalence $(\partial \Delta^{m-1})^{\triangleright} \simeq \Lambda^m_m$ we see that giving a filler of σ is the same as giving a filler of the boundary $\gamma \colon \partial\Delta^{m-1} \to C_{/y}$ with $\gamma(m-1) = f$. By the proof of [Lur18, Tag 05ET], this is the same as solving the lifting problem given by a commutative square

$$\begin{tikzcd}
\partial\Delta^{m-2} \arrow[r] \arrow[d] & (C_{/y})_{/f} \arrow[d] \\
\Delta^{m-2} \arrow[r] & C_{/y}.
\end{tikzcd}$$

Define $(C_{/y})'$ to be the full subcategory of $C_{/y}$ spanned by the objects $g \in C_{/y}$ for which there is a map $g \to f$. Since f is a monomorphism, it is subterminal in $C_{/y}$ by [Lur18, Tag 05EZ] and terminal in $(C_{/y})'$ by [Lur18, Tag 06R3]. By the same proposition, the map $(C_{/y})_{/f} \to (C_{/y})'$ is a trivial Kan fibration (and even an equivalence). Now notice that each vertex of $\partial\Delta^{m-2} \to (C_{/y})_{/f}$ is a map $g \to f$ in $(C_{/y})'$, so that solving our lifting problem is the same as solving that given by the commutative square

$$\begin{tikzcd}
\partial\Delta^{m-2} \arrow[r] \arrow[d] & (C_{/y})_{/f} \arrow[d] \\
\Delta^{m-2} \arrow[r] & (C_{/y})'.
\end{tikzcd}$$

But [Lur18, Tag 0070] implies that the space of such lifts is contractible. $\square$

Proposition 5.7. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Suppose that for each object $C \in \mathcal{C}$ we have an assignment $G(C) \in \mathcal{D}$ and a monomorphism $G(C) \hookrightarrow F(C)$. Suppose furthermore that for each map $C \to C' \in \mathcal{C}$, we have a commutative square*

$$\begin{tikzcd}
G(C) \arrow[r] \arrow[d, hook] & G(C') \arrow[d, hook] \\
F(C) \arrow[r] & F(C')
\end{tikzcd}$$

in $\mathcal{D}$. Then the assignment given by G extends to a functor $G\colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $G \to F$, uniquely up to contractible choice.

Proof. It will suffice to produce, for any $\sigma\colon \Delta^n \to \mathcal{C}$, a prism $\Delta^n \times \Delta^1 \to \mathcal{D}$, restricting to $F(\sigma)$ on $\Delta^1 \times \{1\}$ and to $G(C)$ for each vertex C of σ, uniquely up to contractible choice.

Let

$$\mathrm{Cyl}(n) := (\partial\Delta^n \times \Delta^1) \coprod_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{0\})$$

be the pushout-product of $\{1\} \hookrightarrow \Delta^1$ and $\partial\Delta^n \to \Delta^n$. Let $\mathrm{Cyl}(1) \to \mathcal{D}$ be a cylinder whose bottom face $\Delta^1 \to \mathcal{D}$ is a map $F(C) \to F(C')$ and the top face $\partial\Delta^1 \to \mathcal{D}$ is the pair $G(C), G(C')$. Then the hypotheses of the proposition give a filler $\Delta^1 \times \Delta^1 \to \mathcal{D}$. We will proceed by induction. Fix $n > 1$ and suppose we have prisms $\Delta^{n-1} \times \Delta^1 \to \mathcal{D}$ for any $\sigma\colon \Delta^{n-1} \to \mathcal{C}$. Then for any $\tau\colon \Delta^n \to \mathcal{C}$ we obtain a cylinder $\mathrm{Cyl}(n) \to \mathcal{D}$ from the prisms obtained from the ($(n-1)$-dimensional) faces of τ. By [Lur18, Tag 00TH], there is a chain of simplicial subsets

$$\mathrm{Cyl}(n) =: X(0) \subset X(1) \subset \cdots \subset X(n) \subset X(n+1) := \Delta^n \times \Delta^1,$$

and for $0 \leq i \leq n$, the inclusion map $X(i) \hookrightarrow X(i+1)$ fits into a pushout diagram

$$\begin{tikzcd} \Lambda^{n+1}_{i+1} \arrow[r] \arrow[d] & X(i) \arrow[d] \\ \Delta^{n+1} \arrow[r] & X(i+1). \end{tikzcd}$$

Thus, it suffices to give fillers of the horn inclusions $\Lambda^{n+1}_{i+1} \to \Delta^{n+1}$. These exist and are unique up to contractible space of choices for $i < n$, because $\mathcal{D}$ is an ∞-category. For $i = n$, the proof of [Lur18, Tag 00TH] gives that the last edge of $\Lambda^{n+1}_{n+1} \to \mathcal{D}$ is the monomorphism $G(\tau(n)) \hookrightarrow F(\tau(n))$, so that, since $n+1 \geq 3$, by Proposition 5.6 we also get a contractible space of choices of fillers for the outer horn. $\square$

Construction 5.8. The ∞-category $\mathrm{Shv}(\mathbf{Man})$ is an ∞-topos. Viewed as a monoidal ∞-category with its cartesian product $\times$, it thus admits an enrichment in itself by Remark 4.26. We will denote the resulting $\mathrm{Shv}(\mathbf{Man})^{\times}$-enriched ∞-category by $\mathrm{Shv}(\mathbf{Man})^{M}$. Let $S = \mathrm{Shv}(\mathbf{Man})^{\sim}$ be the core groupoid of $\mathrm{Shv}(\mathbf{Man})$. We denote the categorical algebra corresponding to $\mathrm{Shv}(\mathbf{Man})^{M}$ by

$$M \colon \Delta^{\mathrm{op}}_S \to \mathrm{Shv}(\mathbf{Man})^{\times}.$$

Remark 5.9. The algebra M assigns to every sequence $(\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_{n-1}, \mathcal{X}_n) \in S^{n+1}$ the tuple of mapping stacks

$$(\mathrm{Map}(\mathcal{X}_0, \mathcal{X}_1), \dots, \mathrm{Map}(\mathcal{X}_{n-1}, \mathcal{X}_n)).$$

Proposition 5.10. *Let $(\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_{n-1}, \mathcal{X}_n) \in \Delta^{\mathrm{op}}_S$ be a sequence of stacks. There is a monomorphism*

$$(\mathrm{Emb}(\mathcal{X}_0, \mathcal{X}_1), \dots, \mathrm{Emb}(\mathcal{X}_{n-1}, \mathcal{X}_n)) \hookrightarrow (\mathrm{Map}(\mathcal{X}_0, \mathcal{X}_1), \dots, \mathrm{Map}(\mathcal{X}_{n-1}, \mathcal{X}_n))$$

in $\mathrm{Shv}(\mathbf{Man})^{\times}$.

Proof. This follows directly from the fact that each $\mathrm{Emb}(\mathcal{X}_i, \mathcal{X}_{i+1}) \hookrightarrow \mathrm{Map}(\mathcal{X}_i, \mathcal{X}_{i+1})$ is a monomorphism in $\mathrm{Shv}(\mathbf{Man})$ and that $(\mathrm{Shv}(\mathbf{Man})^{\times})_{[n]} \simeq \mathrm{Shv}(\mathbf{Man})^{n}$. $\square$

Remark 5.11. The embedding substacks are well behaved with respect to the Cartesian symmetric monoidal structure on $\mathrm{Shv}(\mathbf{Man})$. For any stacks $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, we have a commutative square

$$\begin{tikzcd} \mathrm{Emb}(\mathcal{X}, \mathcal{Y}) \times \mathrm{Emb}(\mathcal{Y}, \mathcal{Z}) \arrow[r] \arrow[d, hook] & \mathrm{Emb}(\mathcal{X}, \mathcal{Z}) \arrow[d, hook] \\ \mathrm{Map}(\mathcal{X}, \mathcal{Y}) \times \mathrm{Map}(\mathcal{Y}, \mathcal{Z}) \arrow[r] & \mathrm{Map}(\mathcal{X}, \mathcal{Z}), \end{tikzcd}$$

where the downward maps are monomorphisms. Commutativity follows from the fact that for any manifold T and any fiberwise open substacks $f : \mathcal{X} \times T \to \mathcal{Y}$ and $g : \mathcal{Y} \times T \to \mathcal{Z}$, any composition

$$\mathcal{X} \times T \xrightarrow{(f, \mathrm{pr}_T)} \mathcal{Y} \times T \xrightarrow{g} \mathcal{Z}$$

is again a fiberwise open substack. We have similar diagrams for the unit and projections. More generally, let $S = (\mathrm{Shv}(\mathbf{Man}))^{\sim}$ and consider a map

$$f \colon ([n], (X_0, \dots, X_n)) \to ([m], (Y_0, \dots, Y_m)) \quad \in \quad \Delta^{\mathrm{op}}_S$$

over $\phi\colon [m] \to [n]$ in Δ. We can factor this map via a coCartesian lift of ϕ:

$$([n], (X_0, \ldots, X_n)) \to ([m], (X_{\phi(0)}, \ldots, X_{\phi(m)})) \to ([m], (Y_0, \ldots, Y_m)).$$

For the first factor, we obtain a commutative square

$$\begin{tikzcd}
(\mathrm{Emb}(X_0, X_1), \ldots, \mathrm{Emb}(X_{n-1}, X_n)) \arrow[r] \arrow[d, hook] & (\mathrm{Emb}(X_{\phi(0)}, X_{\phi(1)}), \ldots, \mathrm{Emb}(X_{\phi(m-1)}, X_{\phi(m)})) \arrow[d, hook] \\
(\mathrm{Map}(X_0, X_1), \ldots, \mathrm{Map}(X_{n-1}, X_n)) \arrow[r] & (\mathrm{Map}(X_{\phi(0)}, X_{\phi(1)}), \ldots, \mathrm{Map}(X_{\phi(m-1)}, X_{\phi(m)}))
\end{tikzcd}$$

by the argument above and the fact that $\mathrm{Shv}(\mathbf{Man})^{\times} \to \Delta^{\mathrm{op}}$ is a coCartesian fibration. For the second factor, we get a map

$$(\mathrm{Map}(X_{\phi(0)}, X_{\phi(1)}), \ldots, \mathrm{Map}(X_{\phi(m-1)}, X_{\phi(m)})) \to (\mathrm{Map}(Y_0, Y_1), \ldots, \mathrm{Map}(Y_{m-1}, Y_m))$$

induced by equivalences $X_{\phi(i)} \to Y_i$, which thus also restricts to a map of products of embedding stacks. Choosing a composition, we obtain a commutative diagram

$$\begin{tikzcd}
(\mathrm{Emb}(X_0, X_1), \ldots, \mathrm{Emb}(X_{n-1}, X_n)) \arrow[r] \arrow[d, hook] & (\mathrm{Emb}(Y_0, Y_1), \ldots, \mathrm{Emb}(Y_{m-1}, Y_m)) \arrow[d, hook] \\
(\mathrm{Map}(X_0, X_1), \ldots, \mathrm{Map}(X_{n-1}, X_n)) \arrow[r] & (\mathrm{Map}(Y_0, Y_1), \ldots, \mathrm{Map}(Y_{m-1}, Y_m))
\end{tikzcd}$$

in $\mathrm{Shv}(\mathbf{Man})^{\times}$ for every map f.

Construction 5.12. We now use the description given in Remark 3.19 to enrich $\mathrm{Shv}(\mathbf{Man})$ in embedding stacks. We assign to every sequence $(\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_{n-1}, \mathcal{X}_n)$ the tuple

$$(\mathrm{Emb}(\mathcal{X}_0, \mathcal{X}_1), \ldots, \mathrm{Emb}(\mathcal{X}_{n-1}, \mathcal{X}_n)).$$

After Remark 5.11, we can apply Proposition 5.7 to yield a categorical algebra $E\colon \Delta_S^{\mathrm{op}} \to \mathrm{Shv}(\mathbf{Man})^{\times}$ and a natural transformation $E \to M$.

Proposition 5.13. *The natural transformation $E \to M$ is a monomorphism in* $\mathrm{Alg}_{\mathrm{cat}}(\mathrm{Shv}(\mathbf{Man})^{\times})$.

Proof. We write $\mathcal{V} = \mathrm{Shv}(\mathbf{Man})$. We will check that the diagonal map $E \to E \times_M E$ is an equivalence. We will do so by passing to graphs: the forgetful functor $\mathcal{U}\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Graph}_{\infty}(\mathcal{V})$ preserves limits by Remark 3.42, so we can just check that $U(E) \to U(E) \times_{U(M)} U(E)$ is an equivalence. By Corollary 3.46, we can compute the pullback of graphs $\mathcal{U}(E) \times_{\mathcal{U}(M)} \mathcal{U}(E)$ in the fiber $\mathrm{Fun}(S^2, \mathcal{V})$. Here S is the core of $\mathcal{V}$. For any $(s_1, s_2) \in \mathcal{V}$, we see that $\mathcal{U}(E)(s_1, s_2) = \mathrm{Emb}(s_1, s_2)$ and that $\mathcal{U}(M)(s_1, s_2) = \mathrm{Map}(s_1, s_2)$. Since $\mathrm{Emb}(s_1, s_2) \to \mathrm{Map}(s_1, s_2)$ is a monomorphism in $\mathcal{V}$, the result follows. $\square$

Corollary 5.14. *The categorical algebra $E\colon \Delta_S^{\mathrm{op}} \to \mathrm{Shv}(\mathbf{Man})^{\times}$ is complete, and thus presents a* $\mathrm{Shv}(\mathbf{Man})$*-enriched ∞-category which we denote by* $\mathrm{Shv}(\mathbf{Man})^E$.

Proof. By [GH15, Corollary 5.2.10], E is complete iff the natural map $\iota_0 E \to \iota_1 E$ is an equivalence of spaces. We have a commutative square

$$\begin{tikzcd}
\iota_0 E \arrow[r] \arrow[d] & \iota_1 E \arrow[d] \\
\iota_0 M \arrow[r] & \iota_1 M.
\end{tikzcd}$$

The bottom horizontal map is an equivalence because M is complete. The left downward map is an equivalence by construction: the underlying space of both categorical algebras is the core of Shv(**Man**). The right downward map is by definition $\mathrm{Map}(E^1, E) \to \mathrm{Map}(E^1, M)$, which is a monomorphism of spaces because $E \to M$ is a monomorphism of categorical algebras. Thus the top horizontal map is an equivalence. $\square$

Proposition 5.15. *Restricting to the full subcategory of n-orbifolds gives a functor $\mathbf{Orb}_n^E \to \mathbf{Orb}_n^M$ of* Shv(**Man**)*-enriched ∞-categories. Similarly, restricting to n-manifolds gives a functor $\mathbf{Man}_n^E \to \mathbf{Man}_n^M$. These functors assemble into a commutative diagram of enriched ∞-categories:*

$$\begin{tikzcd}
\mathbf{Man}_n^E \arrow[r] \arrow[d] & \mathbf{Man}_n^M \arrow[d] \\
\mathbf{Orb}_n^E \arrow[r] \arrow[d] & \mathbf{Orb}_n^M \arrow[d] \\
\mathrm{Shv}(\mathbf{Man})^E \arrow[r] & \mathrm{Shv}(\mathbf{Man})^M,
\end{tikzcd}$$

in which the downward functors are fully faithful. Lastly, restricting to orbifold disks gives a commutative diagram

$$\begin{tikzcd}
\mathbf{ODisk}_n^E \arrow[r] \arrow[d] & \mathbf{ODisk}_n^M \arrow[d] \\
\mathbf{Orb}_n^E \arrow[r] & \mathbf{Orb}_n^M
\end{tikzcd}$$

in which the downward maps are fully faithful.

5.3. **Change of enrichment along the shape of a stack.** The goal of this section is to produce honest ∞-categories from our categories enriched in Shv(**Man**). Using that ∞-categories enriched in spaces are just ∞-categories ([GH15, Theorem 5.4.6]), it will suffice to produce a symmetric monoidal functor $\mathrm{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$ that we can use to change the enrichment. There is, of course, a geometric morphism $(\Delta \dashv \Gamma)\colon \mathrm{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$ which sends a sheaf to its space of global sections. But this does not keep much of the structure of the embedding stacks $\mathrm{Emb}(\mathcal{X}, \mathcal{Y})$. For example, for manifolds X and Y, the space of global sections $\Gamma\,\mathrm{Emb}(X, Y)$ is just the *discrete space* of open embeddings from X to Y. We would like to keep the T-fiberwise open embeddings $X \times T \to Y$, which we can think of as T-parameterised *isotopies*. To achieve this, we will use a *shape* functor $\Pi_\infty\colon \mathrm{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$, which we will think of as producing the *homotopy type* of a stack.

Definition 5.16. Let $(\Delta \dashv \Gamma)\colon \mathrm{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$ be the unique geometric morphism of ∞-toposes given by the functor $\Delta\colon \boldsymbol{\mathcal{S}} \to \mathrm{Shv}(\mathbf{Man})$ sending a space to the locally constant stack at that space, and by the functor $\Gamma\colon \mathrm{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$ sending a stack to its global sections: $\Gamma(\mathcal{X}) := \mathcal{X}(*)$.

Definition 5.17. A geometric morphism $(f^* \dashv f_*)\colon \mathcal{E} \to \mathcal{F}$ of ∞-toposes is called an *essential geometric morphism* if the inverse image functor f^* has a left adjoint $f_!\colon \mathcal{E} \to \mathcal{F}$.

Proposition 5.18 ([Clo23, Corollary 6.1.4]). *The geometric morphism*

$$(\Delta \dashv \Gamma)\colon \operatorname{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$$

is essential. We denote the left adjoint to Δ *by* $\Pi_\infty\colon \operatorname{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$ *and call it the* shape *functor.*

Proposition 5.19 ([Clo23, Corollary 6.1.6]). *The shape functor*

$$\Pi_\infty : \operatorname{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$$

preserves finite products.

Definition 5.20. Let $\Delta^\bullet_{\text{ex}}\colon \Delta \to \mathbf{Man}$ be the cosimplicial object of *extended standard simplices* given by

$$[n] \mapsto \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1\} \cong \mathbb{R}^n,$$

with co-degeneracy maps given by addition of consecutive variables, and co-face maps given by insertion of zeros.

Definition 5.21. For a stack $\mathcal{X} \in \operatorname{Shv}(\mathbf{Man})$, we let $\mathcal{X}_\bullet$ denote the simplicial space defined by

$$[n] \mapsto \mathcal{X}(\Delta^n_{\text{ex}}).$$

Proposition 5.22 ([Clo23, Proposition 6.2.2]). *The functor* $\operatorname{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$ *defined by*

$$\mathcal{X} \mapsto \operatorname*{colim}_{\Delta^{\text{op}}} \mathcal{X}_\bullet$$

is equivalent to the shape functor $\Pi_\infty : \operatorname{Shv}(\mathbf{Man}) \to \boldsymbol{\mathcal{S}}$.

Remark 5.23. Since Δ^0_{ex} is the point, Proposition 5.22 gives a natural transformation $\Gamma \Rightarrow \Pi_\infty$ from the global sections functor to the shape functor.

Remark 5.24. Viewing each $\mathcal{X}(\Delta^n_{\text{ex}})$ as a Kan complex, the simplicial space $\mathcal{X}_\bullet$ can be viewed as a bisimplicial set

$$\mathcal{X}_{\bullet,\bullet} : \Delta^{\text{op}} \times \Delta^{\text{op}} \to \mathbf{sSet}$$

defined by the assignment $([n], [m]) \mapsto \mathcal{X}(\Delta^n_{\text{ex}})_m$. Its *diagonal* is the simplicial set $\operatorname{diag}(\mathcal{X}_{\bullet,\bullet})$ given by precomposition with $(\mathrm{id}, \mathrm{id})\colon \Delta^{\text{op}} \to \Delta^{\text{op}} \times \Delta^{\text{op}}$. It is folklore that the diagonal of $\mathcal{X}_{\bullet,\bullet}$ models $\operatorname{colim}_{\Delta^{\text{op}}} \mathcal{X}_\bullet$ as a simplicial set, see for example [Dug08, Example 21.5].

Remark 5.25. Let X and Y be smooth manifolds. In [AFT17, Definition 4.1.2], Ayala, Francis and Tanaka construct a simplicial set $\mathcal{S}\mathrm{nglr}(X, Y)$, which after unraveling the definitions, we see is given by the assignment

$$[n] \mapsto \operatorname{Emb}(X, Y)(\Delta^n_{\text{ex}}),$$

where each discrete space $\operatorname{Emb}(X, Y)(\Delta^n_{\text{ex}})$ is viewed as a set. In [AFT17, Lemma 4.1.4] this simplicial set is shown to be a Kan complex[4]. In particular, we see that we have an equivalence $\Pi_\infty \operatorname{Emb}(X, Y) \simeq \mathcal{S}\mathrm{nglr}(X, Y)$.

[4][AFT17] develops this in the generality of stratified manifolds; we are only concerned with the special case of smooth manifolds.

Proposition 5.26. *The functors* $\Gamma, \Pi_\infty \colon \mathrm{Shv}(\mathbf{Man}) \rightrightarrows \boldsymbol{\mathcal{S}}$ *determine two* change of enrichment *functors*

$$\Gamma_*, (\Pi_\infty)_* \colon \mathbf{Cat}_\infty^{\mathrm{Shv}(\mathbf{Man})} \rightrightarrows \mathbf{Cat}_\infty.$$

The natural transformation $\Gamma \Rightarrow \Pi_\infty$ *induces a natural transformation* $\Gamma_* \Rightarrow (\Pi_\infty)_*$*.*

Proof. The functors Γ and Π_∞ are symmetric monoidal, so by Proposition 3.32 they determine functors

$$\Gamma_*, (\Pi_\infty)_* \colon \mathbf{Cat}_\infty^{\mathrm{Shv}(\mathbf{Man})} \rightrightarrows \mathbf{Cat}_\infty^{\boldsymbol{\mathcal{S}}}.$$

By [GH15, Theorem 5.4.6], the ∞-category of $\boldsymbol{\mathcal{S}}$-enriched ∞-categories $\mathbf{Cat}_\infty^{\boldsymbol{\mathcal{S}}}$ is equivalent to the ∞-category of ∞-categories $\mathbf{Cat}_\infty$. $\square$

The following proposition follows from the fact that the global sections functor Γ sends a mapping stack $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ to the mapping space of $\mathcal{X}, \mathcal{Y}$ in $\mathrm{Shv}(\mathbf{Man})$.

Proposition 5.27. *The change of enrichment functor* Γ_* *applied to* $\mathbf{Man}_n^E$*,* $\mathbf{ODisk}_n^E$*,* $\mathbf{Orb}_n^E$*, and* $\mathrm{Shv}(\mathbf{Man})^E$ *recovers the subcategories* $\mathbf{Man}_n^{oe}$*,* $\mathbf{ODisk}_n^{oe}$*,* $\mathbf{Orb}_n^{oe}$*, and* $\mathrm{Shv}(\mathbf{Man})^{oe}$ *whose objects are those of* $\mathbf{Man}_n$*,* $\mathbf{ODisk}_n$*,* $\mathbf{Orb}_n$*, and* $\mathrm{Shv}(\mathbf{Man})$*, respectively, and whose morphisms are open substacks.*

Definition 5.28. We define the ∞-categories $\mathcal{M}\mathbf{an}_n$, $\mathcal{OD}\mathbf{isk}_n$, $\mathcal{O}\mathbf{rb}_n$ and $\mathcal{S}\mathbf{t}$ to be those obtained by applying the change of enrichment functor $(\Pi_\infty)_*$, defined in Proposition 5.26, to the $\mathrm{Shv}(\mathbf{Man})$-enriched ∞-categories $\mathbf{Man}_n^E$, $\mathbf{ODisk}_n^E$, $\mathbf{Orb}_n^E$, and $\mathrm{Shv}(\mathbf{Man})^E$, respectively.

Corollary 5.29 (of Proposition 5.26)**.** *The natural transformation* $\Gamma_* \Rightarrow (\Pi_\infty)_*$ *yields a diagram of* ∞*-categories*

$$\begin{tikzcd}
\mathbf{Man}_n^{oe} \arrow[r] \arrow[d] & \mathbf{Orb}_n^{oe} \arrow[r] \arrow[d] & \mathrm{Shv}(\mathbf{Man})^{oe} \arrow[d] \\
\mathcal{M}\mathbf{an}_n \arrow[r] & \mathcal{O}\mathbf{rb}_n \arrow[r] & \mathcal{S}\mathbf{t}
\end{tikzcd}$$

for which the horizontal arrows are fully faithful. Considering the inclusion of orbifold disks $\mathbf{ODisk}_n^E \to \mathbf{Orb}_n^E$*, we similarly obtain a commutative square*

$$\begin{tikzcd}
\mathbf{ODisk}_n^{oe} \arrow[r] \arrow[d] & \mathbf{Orb}_n^{oe} \arrow[d] \\
\mathcal{OD}\mathbf{isk}_n \arrow[r] & \mathcal{O}\mathbf{rb}_n
\end{tikzcd}$$

for which the horizontal arrows are fully faithful.

Proof. Commutativity follows from $\Gamma_* \Rightarrow (\Pi_\infty)_*$ being a natural transformation. The horizontal maps are fully faithful because they are fully faithful before change of enrichment by Proposition 5.15 and because fully faithful maps are preserved by change of enrichment, by [GH15, Lemma 5.7.5]. $\square$

In Theorem 7.6 we will use the following lemma to prove that factorization homology is well defined.

Lemma 5.30. *Let $D = \sqcup_{i\in I}[\mathbb{R}^n/G_i]$ be a finite disjoint union of orbifold disks and let $\mathcal{M}, \mathcal{N} \in \mathcal{O}\mathbf{rb}_n$ be n-orbifolds. Given a subset J of I, write $D_J = \sqcup_{j\in J}[\mathbb{R}^n/G_j]$ for the corresponding open suborbifold of D. We have a decomposition*

$$\mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(D, \mathcal{M} \sqcup \mathcal{N}) \simeq \coprod_{I_1 \sqcup I_2 = I} \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(D_{I_1}, \mathcal{M}) \times \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(D_{I_2}, \mathcal{N}).$$

Proof. Recall that mapping spaces between orbifolds come from embedding stacks

$$\mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(D, \mathcal{M} \sqcup \mathcal{N}) := \Pi_\infty \, \mathrm{Emb}(D, \mathcal{M} \sqcup \mathcal{N}),$$

and that the shape functor Π_∞ commutes with colimits and finite products. In particular, by Proposition 5.22, it will suffice to show that we have a decomposition at the level of embeddings

$$\mathrm{Emb}(D, \mathcal{M} \sqcup \mathcal{N})(\Delta^k_{\mathrm{ex}}) \simeq \coprod_{I_1 \sqcup I_2 = I} \mathrm{Emb}(D_{I_1}, \mathcal{M})(\Delta^k_{\mathrm{ex}}) \times \mathrm{Emb}(D_{I_2}, \mathcal{N})(\Delta^k_{\mathrm{ex}}) \tag{1}$$

for each $[k] \in \Delta^{\mathrm{op}}$, for then

$$\begin{aligned} &\operatorname*{colim}_{[k]\in\Delta^{\mathrm{op}}} \mathrm{Emb}(D, \mathcal{M} \sqcup \mathcal{N})(\Delta^k_{\mathrm{ex}}) \\ &\simeq \coprod_{I_1 \sqcup I_2 = I} \operatorname*{colim}_{[k]\in\Delta^{\mathrm{op}}} \mathrm{Emb}(D_{I_1}, \mathcal{M})(\Delta^k_{\mathrm{ex}}) \times \operatorname*{colim}_{[k]\in\Delta^{\mathrm{op}}} \mathrm{Emb}(D_{I_2}, \mathcal{N})(\Delta^k_{\mathrm{ex}}), \end{aligned}$$

as needed. The equivalence in (1) will follow from showing that in the special case $D = [\mathbb{R}^n/G]$, any map $D \times \Delta^k_{\mathrm{ex}} \to \mathcal{M} \sqcup \mathcal{N}$ factors via one of $\mathcal{M} \hookrightarrow \mathcal{M} \sqcup \mathcal{N}$ or $\mathcal{N} \hookrightarrow \mathcal{M} \sqcup \mathcal{N}$. This is a standard fact about stacks, which we spell out. Consider the pullback squares

$$\begin{tikzcd} P_{\mathcal{M}} := (\mathbb{R}^n \times \Delta^k_{\mathrm{ex}}) \times_{\mathcal{M}\sqcup\mathcal{N}} (\mathcal{M}) \arrow[r] \arrow[d] & D_{\mathcal{M}} := (D \times \Delta^k_{\mathrm{ex}}) \times_{\mathcal{M}\sqcup\mathcal{N}} \mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathbb{R}^n \times \Delta^k_{\mathrm{ex}} \arrow[r] & D \times \Delta^k_{\mathrm{ex}} \arrow[r] & \mathcal{M} \sqcup \mathcal{N}, \end{tikzcd}$$

where $\mathbb{R}^n \times \Delta^k_{\mathrm{ex}} \to D \times \Delta^k_{\mathrm{ex}}$ is an atlas. Since the inclusion $\mathcal{M} \to \mathcal{M} \sqcup \mathcal{N}$ is an open and closed substack, the map of manifolds $P_{\mathcal{M}} \to \mathbb{R}^n \times \Delta^k_{\mathrm{ex}}$ is an open and closed embedding. In particular, $P_{\mathcal{M}}$ is either all of $\mathbb{R}^n \times \Delta^k_{\mathrm{ex}}$ or it is empty. Without loss of generality, assume it is empty (then $P_{\mathcal{N}}$ is all of $\mathbb{R}^n \times \Delta^k_{\mathrm{ex}}$). Then $\emptyset \to D_{\mathcal{M}}$ is an atlas, and this implies $D_{\mathcal{M}}$ is empty. Since, by universality of colimits, we have the decomposition $D \times \Delta^k_{\mathrm{ex}} \simeq D_{\mathcal{M}} \sqcup D_{\mathcal{N}}$, we are done. □

Definition 5.31. Let the ∞-category $\mathbf{Disk}_n$ be the full subcategory of $\mathrm{Shv}(\mathbf{Man})$ whose objects are finite disjoint unions of n-dimensional Euclidean spaces. Applying the constructions of this section to $\mathbf{Disk}_n \hookrightarrow \mathbf{Man}$, we obtain the full subcategories $\mathbf{Disk}_n^{oe} \hookrightarrow \mathbf{Man}_n^{oe} \hookrightarrow \mathbf{Orb}_n^{oe}$ and $\mathcal{D}\mathbf{isk}_n \hookrightarrow \mathcal{M}\mathbf{an}_n \hookrightarrow \mathcal{O}\mathbf{rb}_n$, together with the functors $\mathbf{Disk}_n^{oe} \to \mathcal{D}\mathbf{isk}_n$ and $\mathbf{Man}_n^{oe} \to \mathcal{M}\mathbf{an}_n$.

Remark 5.32. By Remark 5.25, our definition of the ∞-categories $\mathcal{D}\mathbf{isk}_n \hookrightarrow \mathcal{M}\mathrm{an}_n$ is equivalent to the definition given in [AFT17, Convention 4.1.5] when restricted from stratified spaces to smooth manifolds.

5.4. **Disks over the closed interval.**

Definition 5.33. Let $\mathbf{Man}^\partial$ be the category of smooth manifolds with boundary.

Remark 5.34. There is a fully faithful inclusion $\mathbf{Man} \hookrightarrow \mathbf{Man}^\partial$ which induces a restriction functor $\mathrm{Shv}(\mathbf{Man}^\partial) \to \mathrm{Shv}(\mathbf{Man})$.

Definition 5.35. Since the topology of open covers on smooth manifolds with boundary is subcanonical, we view the closed interval I and the half-open endpoints $[-1, 1)$ and $(-1, 1]$ as stacks on smooth manifolds via the composition

$$\mathbf{Man}^\partial \to \mathrm{Shv}(\mathbf{Man}^\partial) \to \mathrm{Shv}(\mathbf{Man}).$$

Remark 5.36. An open embedding $U \to I$ in $\mathbf{Man}^\partial$ becomes an open substack in $\mathrm{Shv}(\mathbf{Man})$. Indeed, for any $M \in \mathbf{Man}$ and a map $M \to I \in \mathrm{Shv}(\mathbf{Man})$ from a manifold M, the pullback $U \times_I M \to M$ is a map in $\mathbf{Man}^\partial$ because the composition $\mathbf{Man}^\partial \to \mathrm{Shv}(\mathbf{Man})$ preserves limits, and pullbacks of open embeddings of manifolds with boundary are open embeddings.

Definition 5.37. Let $\mathbf{Disk}_1^\partial$ be the full ∞-subcategory of $\mathbf{Man}^\partial$ spanned by finite disjoint unions of the disks $[-1, 1)$, $\mathbb{R}$ and $(-1, 1]$.

We let $\mathbf{Disk}_1^{\partial,oe}$ be the ∞-subcategory with the same objects and maps the open embeddings, this corresponds to [AFT16, Definition 2.2]. We let $\mathcal{D}\mathbf{isk}_1^\partial$ be the ∞-category of disks as defined in [AFT16, Notation 2.8]. The construction is analogous to our construction of $\mathcal{D}\mathbf{isk}_n$ in the case of manifolds with boundary: the objects of $\mathcal{D}\mathbf{isk}_1^\partial$ are those of $\mathbf{Disk}_1^{oe}$ and there is a functor of ∞-categories

$$\mathbf{Disk}_1^{\partial,oe} \to \mathcal{D}\mathbf{isk}_1^\partial.$$

6. Symmetric monoidal structure via disjoint union

The goal of this section is to endow the ∞-category of n-orbifolds $\mathcal{O}\mathbf{rb}_n$ with a symmetric monoidal structure given by the disjoint union of $\mathrm{Shv}(\mathbf{Man})$. We will construct these symmetric monoidal structures for the enriched categories constructed in Section 5.2.

Proposition 6.1. *The cartesian symmetric monoidal structure on* $\mathrm{Shv}(\mathbf{Man})$ *induces a cartesian symmetric monoidal structure on the* ∞*-category of* $\mathrm{Shv}(\mathbf{Man})$*-enriched* ∞*-categories* $\mathbf{Cat}_\infty^{\mathrm{Shv}(\mathbf{Man})}$*. Moreover, the change of enrichment functor*

$$(\Pi_\infty)_* \colon \mathbf{Cat}_\infty^{\mathrm{Shv}(\mathbf{Man})} \to \mathbf{Cat}_\infty$$

upgrades to a lax symmetric monoidal functor with respect to the cartesian symmetric monoidal structures.

Proof. The first statement follows immediately from Corollary 3.39, the second from Proposition 3.40. □

6.1. **Commutative algebra objects of stack-enriched categories.** We will show that coproducts in $\mathrm{Shv}(\mathbf{Man})$ can be used to yield a commutative algebra object of $\mathbf{Cat}_\infty^{\mathrm{Shv}(\mathbf{Man})}$ when this ∞-category is endowed with the Cartesian symmetric monoidal structure (Corollary 6.11).

For this subsection, we fix an ∞-topos $\mathcal{V}$, and we let $\mathbf{Cat}_\infty^K$ be the subcategory of $\mathbf{Cat}_\infty$ spanned by the ∞-categories that admit finite coproducts and by the functors that preserve finite coproducts.

Proposition 6.2. *The ∞-category $\mathbf{Cat}_\infty^K$ admits a closed symmetric monoidal structure $(\mathbf{Cat}_\infty^K)^\otimes$ with respect to which the inclusion*

$$\mathbf{Cat}_\infty^K \hookrightarrow \mathbf{Cat}_\infty$$

is lax symmetric monoidal, where $\mathbf{Cat}_\infty$ has the usual Cartesian symmetric monoidal structure.

Proof. By [Lur, Corollary 4.8.1.4] $\mathbf{Cat}_\infty^K$ has a symmetric monoidal structure compatible with the inclusion and by [Lur, Remark 4.8.1.6] it is closed. $\square$

Lemma 6.3. *The inclusion $\mathbf{Cat}_\infty^K \hookrightarrow \mathbf{Cat}_\infty$ preserves and creates products.*

Intuitively, the tensor product $\mathcal{C} \otimes \mathcal{D}$ of $\mathbf{Cat}_\infty^K$ can be described as follows: for any $\mathcal{E} \in \mathbf{Cat}_\infty^K$, the maps $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ correspond to the maps $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that preserve coproducts in both variables.

Definition 6.4 ([GGN15, Definition 2.1])**.** An ∞-category $\mathcal{C}$ is preadditive if it is pointed (admits a zero object), admits finite coproducts and finite products, and the canonical morphism $C_1 \sqcup C_2 \to C_1 \times C_2$ is an equivalence for all objects C_1, C_2 in $\mathcal{C}$.

Proposition 6.5. *The ∞-category $\mathbf{Cat}_\infty^K$ is preadditive.*

Proof. See the proof of [GGN15, Theorem 8.8]. $\square$

The ∞-topos $\mathcal{V}$ gives an associative algebra object of $(\mathbf{Cat}_\infty^K)^\otimes$ via Cartesian product, again by the proof of [GGN15, Theorem 8.8]. We denote it by $(\mathcal{V}, \times) \in \mathrm{Alg}(\mathbf{Cat}_\infty^K)$.

Proposition 6.6. *The forgetful functor $\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K) \to \mathbf{Cat}_\infty^K$ sending a module to its underlying ∞-category preserves and creates finite products and coproducts. In particular, $\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K)$ is preadditive, and thus the induced Cartesian and coCartesian symmetric monoidal structures on $\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K)$ agree.*

Proof. The forgetful functor $U\colon \mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K) \to \mathbf{Cat}_\infty^K$ preserves and creates finite products by [Lur, Corollary 4.2.3.3]. Since the symmetric monoidal structure of $\mathbf{Cat}_\infty^K$ is closed, U also preserves and creates finite coproducts by [Lur, Corollary 4.2.3.5]. $\square$

Corollary 6.7. *There is an equivalence*

$$\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K) \simeq \mathrm{CAlg}(\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K)^\times),$$

where on the right-hand side the commutative algebra objects are with respect to the Cartesian symmetric monoidal structure on $\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K)$.

Proof. The Cartesian symmetric monoidal structure on $\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K)$ agrees with the coCartesian symmetric monoidal structure. Now the equivalence is that of [Lur, Corollary 2.4.3.10], which is implemented by the map constructed in [Lur, Example 2.4.3.5]. $\square$

Remark 6.8. By naturality of the construction in [Lur, Example 2.4.3.5], the equivalence of Corollary 6.7 sits in a commutative diagram

$$\begin{tikzcd} \mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K) \arrow[r, "\simeq"'] \arrow[d, "U"] & \mathrm{CAlg}(\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K)) \arrow[d, "\mathrm{CAlg}(U)"] \\ \mathbf{Cat}_\infty^K \arrow[r, "\simeq"'] & \mathrm{CAlg}(\mathbf{Cat}_\infty^K). \end{tikzcd}$$

Corollary 6.9. *The map* $\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K) \to \mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty)$ *preserves products, and is thus a symmetric monoidal functor with respect to the Cartesian symmetric monoidal structures.*

Combining the last two corollaries yields a commutative diagram

$$\begin{tikzcd} \mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty^K) \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathrm{LMod}_{(\mathcal{V},\times)}(\mathbf{Cat}_\infty)^\times) \arrow[d] \\ \mathbf{Cat}_\infty^K \arrow[r] & \mathrm{CAlg}(\mathbf{Cat}_\infty). \end{tikzcd}$$

On the top left-hand side, the algebra $(\mathcal{V}, \times)$ gives $\mathcal{V}$ as a left module over itself. The bottom horizontal arrow, induced by the preadditivity of $\mathbf{Cat}_\infty^K$, sends $\mathcal{V}$ to the commutative algebra object $(\mathcal{V}, \sqcup) \in \mathrm{CAlg}(\mathbf{Cat}_\infty)$.

Specialising to the ∞-topos $\mathcal{V} = \mathrm{Shv}(\mathbf{Man})$, we obtain the following corollary.

Corollary 6.10. *The functor* $\mathrm{Shv}(\mathbf{Man})^\sqcup \colon \mathbf{Fin}_* \to \mathbf{Cat}_\infty$ *lifts along the forgetful functor*

$$\mathrm{LMod}_{\mathrm{Shv}(\mathbf{Man})^\times}(\mathbf{Cat}_\infty) \to \mathbf{Cat}_\infty$$

to yield a commutative diagram

$$\begin{tikzcd} & \mathrm{LMod}_{\mathrm{Shv}(\mathbf{Man})^\times}(\mathbf{Cat}_\infty) \arrow[d] \\ \mathbf{Fin}_* \arrow[ur, dashed, "\Phi"] \arrow[r, "\mathrm{Shv}(\mathbf{Man})^\sqcup"'] & \mathbf{Cat}_\infty, \end{tikzcd}$$

where Φ *sends coproducts to products and thus gives a commutative algebra object of* $\mathrm{LMod}_{(\mathrm{Shv}(\mathbf{Man}))}(\mathbf{Cat}_\infty)$.

Corollary 6.11. *The* $\mathrm{Shv}(\mathbf{Man})$*-enriched* ∞*-category* $\mathrm{Shv}(\mathbf{Man})^M$ *gives a commutative algebra object of* $(\mathbf{Cat}_\infty^{\mathrm{Shv}(\mathbf{Man})})^\times$*, with the tensor product functor induced from disjoint union of stacks.*

Proof. Apply Corollary 4.24 to the commutative algebra object obtained in Corollary 6.10. $\square$

6.2. **Symmetric monoidal structure on orbifolds.** We now restrict the symmetric monoidal structure on $\mathrm{Shv}(\mathbf{Man})^M$ induced by coproducts to a symmetric monoidal structure on $\mathrm{Shv}(\mathbf{Man})^E$ and use this to produce a symmetric monoidal structure on $\mathcal{O}\mathbf{rb}_n$.

Remark 6.12. Embedding stacks are well behaved with respect to disjoint union in $\mathrm{Shv}(\mathbf{Man})$. Specifically, given any stacks X_0, X_1, Y_0, Y_1, we have a pullback square

$$\begin{tikzcd}
\mathrm{Emb}(X_0, Y_0) \times \mathrm{Emb}(X_1, Y_1) \arrow[r, hook] \arrow[d, hook] & \mathrm{Emb}(X_0 \sqcup X_1, Y_0 \sqcup Y_1) \arrow[d, hook] \\
\mathrm{Map}(X_0, Y_0) \times \mathrm{Map}(X_1, Y_1) \arrow[r, hook, "\sqcup"] & \mathrm{Map}(X_0 \sqcup X_1, Y_0 \sqcup Y_1).
\end{tikzcd}$$

Note that all the arrows in this pullback square are monomorphisms in Shv(**Man**).

Remark 6.13. Consider the functor $\sqcup\colon M \times M \to M$ obtained from Corollary 6.11. We have a monomorphism $E \to M$ by Proposition 5.13, and thus a monomorphism on products $E \times E \to M \times M$. We would like to show that this functor restricts to a functor $\sqcup\colon E \times E \to E$ along the monomorphisms such that the diagram

$$\begin{tikzcd}
E \times E \arrow[r, dashed, "\sqcup"] \arrow[d, hook] & E \arrow[d, hook] \\
M \times M \arrow[r, "\sqcup"] & M
\end{tikzcd}$$

commutes. Consider the pullback $P = (E \times E) \times_M E$ of $E \hookrightarrow M$ along

$$E \times E \to M \times M \xrightarrow{\sqcup} M.$$

Providing a lift $E \times E \to M \times M$ is the same as providing a section of the projection $p\colon P \hookrightarrow E \times E$. This is in turn the same as showing that p is an equivalence, because p is a monomorphism. Write $\mathcal{V} = \mathrm{Shv}(\mathbf{Man})$ and S for the core groupoid of Shv(**Man**). Since the forgetful functor $U\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Graph}_\infty(\mathcal{V})$ is conservative on fibers, we need to show that $U(p)$ is an equivalence in the pullback square

$$\begin{tikzcd}
U(P) \arrow[rr] \arrow[d, "U(p)"'] & & U(E) \arrow[d] \\
U(E) \times U(E) \arrow[r] & U(M) \times U(M) \arrow[r, "U(\sqcup)"'] & U(M).
\end{tikzcd}$$

We unpack the definitions. The map $U(E) \times U(E) \to U(M) \times U(M)$ is the natural transformation between functors

$$\mathrm{Emb}(-,-) \times \mathrm{Emb}(-,-), \mathrm{Map}(-,-) \times \mathrm{Map}(-,-)\colon (S \times S)^2 \rightrightarrows \mathcal{V}$$

given by the substack inclusions. The map $U(M) \times U(M) \xrightarrow{U(\sqcup)} U(M)$ is the transformation from $\mathrm{Map}(-,-) \times \mathrm{Map}(-,-)$ to the composite functor

$$(S \times S)^2 \xrightarrow{F} S \times S \xrightarrow{\mathrm{Map}(-,-)} \mathcal{V}.$$

Here $F : (S \times S)^2 \to S \times S$ is defined by the assignment

$$(s, t), (s', t') \mapsto (s \sqcup s', t \sqcup t').$$

By Corollary 3.46 it will suffice to show that the objectwise maps

$$P((s, t), (s', t')) \to U(E)(s, t) \times U(E)(s', t')$$

are equivalences. By Remark 3.45 functor $U(p)\colon (S \times S)^2 \to \mathcal{V}$ is computed by the assignment

$$(s, t), (s', t') \mapsto (U(E)(s, t) \times U(E)(s', t')) \times_{U(M)(s \sqcup s', t \sqcup t')} U(E)(s \sqcup s', t \sqcup t').$$

Expanding out notation, this objectwise pullback is

$$(\mathrm{Emb}(s,t) \times \mathrm{Emb}(s',t')) \times_{\mathrm{Map}(s,t)\times \mathrm{Map}(s',t')} \mathrm{Emb}(s \sqcup s', t \sqcup t').$$

We now see that the projection to $\mathrm{Emb}(s,t) \times \mathrm{Emb}(s',t')$ is an equivalence by Remark 6.12.

Proposition 6.14. *The symmetric monoidal structure on M given by disjoint union in* $\mathrm{Shv}(\mathbf{Man})$ *restricts along $E \hookrightarrow M$ to a commutative algebra object of* $(\mathbf{Cat}_\infty^{\mathrm{Shv}(\mathbf{Man})})^\times$ *with underlying object E.*

Proof. Denote by $M^\sqcup \colon \mathbf{Fin}_* \to \mathbf{Cat}_\infty^{\mathrm{Shv}(\mathbf{Man})}$ the functor corresponding to the symmetric monoidal structure on M given by disjoint union. For any map $\alpha\colon \langle n\rangle \to \langle m\rangle \in \mathbf{Fin}_*$, the argument given in Remark 6.13 generalises to restrict the map $M^\sqcup(\alpha)\colon M^n \to M^m$ to a map $E^n \to E^m$. Now apply Proposition 5.7. □

Proposition 6.15. *Disjoint union determines symmetric monoidal structures on $\mathcal{M}\mathbf{an}_n$, $\mathcal{O}\mathcal{D}\mathbf{isk}_n$, $\mathcal{O}\mathbf{rb}_n$ and $\mathcal{S}\mathbf{t}$ compatible with the inclusions.*

Proof. This follows immediately from Proposition 6.1 applied to the symmetric monoidal $\mathrm{Shv}(\mathbf{Man})$-enriched ∞-category obtained in Proposition 6.14. □

Remark 6.16. The empty stack $\emptyset$ is the monoidal unit in each of the symmetric monoidal ∞-categories of Proposition 6.15, and it is initial in each.

Definition 6.17. By [AFT16, Notation 1.21], the inclusion of symmetric monoidal categories $\mathcal{O}\mathcal{D}\mathbf{isk}_n \to \mathcal{O}\mathbf{rb}_n$ makes the slice category $\mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}}$ into an ∞-operad for any orbifold $\mathcal{M}$. We will denote this ∞-operad by $(\mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}})^\otimes$. Informally, we can describe this ∞-operad as having objects of $\mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}}$, and operations $(y_i \to x)_{i\in I} \to (z \to x)$ given by a map $\sqcup_i y_i \to x$ and a map $(\sqcup_i y_i \to x) \to (z \to x)$ in the slice (c.f. [KSW25, Example A.4.6]). Similarly, the inclusion $\mathbf{ODisk}_n^{oe} \to \mathbf{Orb}_n^{oe}$ makes the slice category $\mathbf{ODisk}_{n/\mathcal{M}}^{oe}$ into an ∞-operad $(\mathbf{ODisk}_{n/\mathcal{M}}^{oe})^\otimes$. We also have ∞-operads $(\mathcal{O}\mathbf{rb}_{n/\mathcal{M}})^\otimes$ and $(\mathbf{Orb}_{n/\mathcal{M}}^{oe})^\otimes$.

Remark 6.18. Again by [AFT16, Notation 1.21], the functors $\mathbf{ODisk}_n^{oe} \to \mathcal{O}\mathcal{D}\mathbf{isk}_n$ and $\mathbf{Orb}_n^{oe} \to \mathcal{O}\mathbf{rb}_n$ induce maps of ∞-operads

$$(\mathbf{Orb}_{n/\mathcal{M}}^{oe})^\otimes \to (\mathcal{O}\mathbf{rb}_{n/\mathcal{M}})^\otimes \text{ and } (\mathbf{ODisk}_{n/\mathcal{M}}^{oe})^\otimes \to (\mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}})^\otimes.$$

Similarly, the four forgetful maps of the form $\mathcal{O}_{/\mathcal{M}} \to \mathcal{O}$ upgrade to maps of ∞-operads.

Remark 6.19. In the case of disks over the closed interval defined in Definition 5.37, by [AFT16, Notation 1.21] we also have ∞-operads $(\mathbf{Disk}_{1/I}^{\partial,oe})^\otimes$ and $(\mathcal{D}\mathbf{isk}_{1/I}^{\partial})^\otimes$.

Remark 6.20. [AFT16, Proposition 2.22] says that the ∞-operad $(\mathcal{D}\mathbf{isk}_{1/I}^{\partial})^\otimes$ arises as an ∞-operadic localization of the ∞-operad $(\mathbf{Disk}_{1/I}^{\partial,oe})^\otimes$ at the maps $(U \hookrightarrow I) \to (V \hookrightarrow I) \in \mathbf{Disk}_{1/I}^{\partial,oe}$ that become equivalences in $\mathcal{D}\mathbf{isk}_{1/I}^{\partial}$ (these are referred to as *isotopy equivalences*). In particular, for each symmetric monoidal ∞-category $\mathcal{V}$, restriction defines a fully faithful functor

$$\mathrm{Alg}_{\mathcal{D}\mathbf{isk}_{1/I}^{\partial}}(\mathcal{V}) \hookrightarrow \mathrm{Alg}_{\mathbf{Disk}_{1/I}^{\partial,oe}}(V)$$

whose image consists of those $\mathbf{Disk}_{1/I}^{\partial,oe}$ algebras that carry isotopy equivalences to equivalences in $\mathcal{V}$.

There is, however, a gap in the proof given in [AFT16]. This is addressed in [KSW25, Theorem 5.2.3] for smooth conical manifolds and in [Ara26, Theorem A.1] for smooth manifolds.

7. Factorization homology of orbifolds

For this section, we fix a symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$. We now have enough background to define factorization homology for orbifolds and view it as a left Kan extension. The main result of this section is that given an algebra $A\colon \mathcal{O}\mathcal{D}\mathbf{isk}_n \to \mathcal{V}^{\otimes}$, we obtain a functor

$$\int_{-} A\colon \mathcal{O}\mathbf{rb}_n \to \mathcal{V}^{\otimes}$$

that is symmetric monoidal.

Definition 7.1. The ∞-category of $\mathcal{O}\mathcal{D}\mathbf{isk}_n$-algebras in $\mathcal{V}^{\otimes}$

$$\mathrm{Alg}_{\mathcal{O}\mathcal{D}\mathbf{isk}_n}(\mathcal{V}^{\otimes}) := \mathrm{Fun}^{\otimes}(\mathcal{O}\mathcal{D}\mathbf{isk}_n^{\sqcup}, \mathcal{V}^{\otimes})$$

is the ∞-category of symmetric monoidal functors.

Definition 7.2. Let $\mathcal{X} \in \mathcal{O}\mathbf{rb}_n$ be an n-orbifold. Let $A\colon \mathcal{O}\mathcal{D}\mathbf{isk}_n^{\sqcup} \to \mathcal{V}^{\otimes}$ be an $\mathcal{O}\mathcal{D}\mathbf{isk}_n$-algebra in $\mathcal{V}^{\otimes}$. *Factorization homology (of $\mathcal{X}$ with coefficients in A)* is an object of $\mathcal{V}$ given by the expression

$$\int_{\mathcal{X}} A := \mathrm{colim}(\mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{X}} \to \mathcal{O}\mathcal{D}\mathbf{isk}_n \xrightarrow{A} \mathcal{V}),$$

provided this colimit exists.

Remark 7.3. It follows immediately from Remark 5.32 that Definition 7.2 specialises to the factorization homology of smooth manifolds given in [AFT16, Theorem 2.15].

Definition 7.4 ([AF15, Definition 3.4])**.** A symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$ is $\otimes$-presentable if it satisfies the following conditions.

- $\mathcal{V}^{\otimes}$ is presentable: with respect to an understood fixed uncountable cardinal, $\mathcal{V}$ admits colimits and every object is a filtered colimit of compact objects.
- The monoidal structure distributes over small colimits: for each object $V \in \mathcal{V}$, the functor $V \otimes -\colon \mathcal{V} \to \mathcal{V}$ carries colimit diagrams to colimit diagrams.

Suppose $\mathcal{V}^{\otimes}$ is $\otimes$-presentable and consider the symmetric monoidal functor $\iota\colon \mathcal{O}\mathcal{D}\mathbf{isk}_n \to \mathcal{O}\mathbf{rb}_n$.

Remark 7.5. The precomposition functor $\iota^*\colon \mathrm{Fun}(\mathcal{O}\mathbf{rb}_n, \mathcal{V}) \to \mathrm{Fun}(\mathcal{O}\mathcal{D}\mathbf{isk}_n, \mathcal{V})$ has a left adjoint

$$\iota_!\colon \mathrm{Fun}(\mathcal{O}\mathcal{D}\mathbf{isk}_n, \mathcal{V}) \to \mathrm{Fun}(\mathcal{O}\mathbf{rb}_n, \mathcal{V})$$

defined by left Kan extension along ι. For an algebra $A \in \mathrm{Alg}_{\mathcal{O}\mathcal{D}\mathbf{isk}_n}(\mathcal{V}^{\otimes})$, we have

$$\iota_! A(X) \simeq \int_X A \in \mathrm{Fun}(\mathcal{O}\mathbf{rb}_n, \mathcal{V}).$$

Theorem 7.6. *Suppose $\mathcal{V}^{\otimes}$ is $\otimes$-presentable. The adjunction $\iota_! \dashv \iota^*$ restricts to an adjunction between symmetric monoidal categories*

$$\iota_!^{\otimes}\colon \mathrm{Alg}_{\mathcal{O}\mathcal{D}\mathbf{isk}_n}(\mathcal{V}^{\otimes}) \rightleftarrows \mathrm{Fun}^{\otimes}(\mathcal{O}\mathbf{rb}_n^{\sqcup}, \mathcal{V}^{\otimes})\colon \iota^*.$$

Proof. The result will follow from [AFT16, Lemma 2.16] once we show that for any n-orbifolds $\mathcal{M}, \mathcal{N}$, the functor

$$F\colon \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}} \times \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{N}} \to \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}\sqcup\mathcal{N}}$$

is cofinal; and that the functor

$$G\colon \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathbf{1}_{\mathcal{O}\mathcal{D}\mathbf{isk}_n}} \to \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathbf{1}_{\mathcal{O}\mathbf{rb}_n}}$$

is cofinal. The functor G is an equivalence; the units are both the empty manifold. We think the functor F is also an equivalence, but we will only prove cofinality here. We fix notation $\mathcal{O}_{\mathcal{M}} := \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}}$, $\mathcal{O}_{\mathcal{N}} := \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{N}}$, and $\mathcal{O}_{\mathcal{M}\sqcup\mathcal{N}} = \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}\sqcup\mathcal{N}}$. It will suffice to show that given any $(d\colon D \to \mathcal{M} \sqcup \mathcal{N}) \in \mathcal{O}_{\mathcal{M}\sqcup\mathcal{N}}$, the ∞-category

$$C := (\mathcal{O}_{\mathcal{M}} \times \mathcal{O}_{\mathcal{N}}) \times_{\mathcal{O}_{\mathcal{M}\sqcup\mathcal{N}}} (\mathcal{O}_{\mathcal{M}\sqcup\mathcal{N}})_{d/}$$

is weakly contractible. The classifying space of C can be expressed as the colimit

$$B(C) \simeq \underset{\substack{(f\colon O \to \mathcal{M}) \in \mathcal{O}_{\mathcal{M}},\\ (g\colon O' \to \mathcal{N}) \in \mathcal{O}_{\mathcal{N}}}}{\mathrm{colim}} \mathrm{Map}_{\mathcal{O}_{\mathcal{M}\sqcup\mathcal{N}}}(d, F(f,g)).$$

Consider now the fiber sequence of mapping spaces

$$\mathrm{Map}_{\mathcal{O}_{\mathcal{M}\sqcup\mathcal{N}}}(d, F(f,g)) \to \mathrm{Map}_{\mathcal{O}\mathcal{D}\mathbf{isk}_n}(D, O \sqcup O') \xrightarrow{(f,g)\circ -} \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(D, \mathcal{M} \sqcup \mathcal{N}),$$

where the fiber is taken at $d \in \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(D, \mathcal{M} \sqcup \mathcal{N})$. Write $D = \sqcup_{i \in I}[\mathbb{R}^n/G_i]$. By Lemma 5.30, we can decompose mapping spaces in the second map, yielding

$$\begin{gathered}\textstyle\coprod_{I_1 \sqcup I_2} \mathrm{Map}_{\mathcal{O}\mathcal{D}\mathbf{isk}_n}(D_{I_1}, O) \times \mathrm{Map}_{\mathcal{O}\mathcal{D}\mathbf{isk}_n}(D_{I_2}, O')\\ \big\downarrow \\ \textstyle\coprod_{I_1 \sqcup I_2} \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(D_{I_1}, \mathcal{M}) \times \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(D_{I_2}, \mathcal{N}).\end{gathered}$$

The homotopy fiber of d will thus be

$$\mathrm{Map}_{\mathcal{O}_{\mathcal{M}\sqcup\mathcal{N}}}(d, F(f,g)) \simeq \mathrm{Map}_{\mathcal{O}_{\mathcal{M}}}(D_{I_1} \xrightarrow{d_1} \mathcal{M}, O \xrightarrow{f} \mathcal{M}) \times \mathrm{Map}_{\mathcal{O}_{\mathcal{N}}}(D_{I_2} \xrightarrow{d_2} \mathcal{N}, O' \xrightarrow{g} \mathcal{N}),$$

for some decomposition $I_1 \sqcup I_2 = I$. Therefore the colimit expression for the classifying space becomes

$$\begin{aligned} B(C) &\simeq \underset{(f\colon O \to \mathcal{M}) \in \mathcal{O}_{\mathcal{M}}}{\mathrm{colim}} \; \underset{(g\colon O' \to \mathcal{N}) \in \mathcal{O}_{\mathcal{N}}}{\mathrm{colim}} \mathrm{Map}_{\mathcal{O}_{\mathcal{M}}}(d_1, f) \times \mathrm{Map}_{\mathcal{O}_{\mathcal{N}}}(d_2, g) \\ &\simeq B((\mathcal{O}_{\mathcal{M}})_{d_1/}) \times B((\mathcal{O}_{\mathcal{N}})_{d_2/}). \end{aligned}$$

But both $(\mathcal{O}_{\mathcal{M}})_{d_1/}$ and $(\mathcal{O}_{\mathcal{N}})_{d_2/}$ have initial objects and thus contractible classifying spaces. $\square$

8. Excision

Definition 8.1. A map $f\colon \mathcal{M} \to I$ in $\mathrm{Shv}(\mathbf{Man})$ from an orbifold $\mathcal{M}$ to the closed interval I is a *collar-gluing of* $\mathcal{M}$ if its restriction along $(-1,1) \to I$ is isomorphic to the projection $(-1,1) \times \mathcal{M}_0 \to (-1,1)$ for some orbifold $\mathcal{M}_0$.

The following construction and proposition are counterparts to [AFT16, Lemma 2.24].

Construction 8.2. Let $f\colon \mathcal{M} \to I$ be a collar-gluing with $\mathcal{M}$ an n-orbifold. We will define a functor of ∞-operads

$$f^{-1}\colon (\mathbf{Disk}^{\partial,oe}_{1/I})^{\otimes} \to (\mathbf{Orb}^{oe}_{n/\mathcal{M}})^{\otimes} \to (\mathcal{O}\mathbf{rb}_{n/\mathcal{M}})^{\otimes}.$$

The second functor is described in Remark 6.18. We claim that for any finite disjoint union of disks $U \hookrightarrow I$, the pullback $(U \times_I \mathcal{M} \to \mathcal{M}) \in \mathrm{Shv}(\mathbf{Man})$ is an open substack. Indeed, for any map $T \to \mathcal{M}$ from a manifold T, the pullback $(U \times_I \mathcal{M}) \times_{\mathcal{M}} T$ is just $U \times_I T$, and since the functor $\mathbf{Man}^{\partial} \to \mathrm{Shv}(\mathbf{Man})$ preserves limits, this is just the pullback in $\mathbf{Man}^{\partial}$, which exists because $U \to I$ is an open embedding. The assignment

$$(U \to I) \mapsto (U \times_I \mathcal{M} \to \mathcal{M})$$

thus gives a functor $\mathbf{Disk}^{\partial,oe}_{1/I} \to \mathbf{Orb}^{oe}_{n/\mathcal{M}}$ of ∞-categories, which upgrades via Remark 6.18 to one on the operads, giving the first functor.

Proposition 8.3. *In the setup of Construction 8.2, the ∞-operad map f^{-1} extends to a functor of ∞-operads $f^{-1}\colon (\mathcal{D}\mathbf{isk}^{\partial}_{1/I})^{\otimes} \to (\mathcal{O}\mathbf{rb}_{n/\mathcal{M}})^{\otimes}$.*

Proof. By Remark 6.20 it will suffice to show that maps $(U \hookrightarrow I) \hookrightarrow (V \hookrightarrow I)$ in $\mathbf{Disk}^{\partial,oe}_{1/I}$ that become equivalences in $\mathcal{D}\mathbf{isk}^{\partial}_{1/I}$ also become equivalences under f^{-1}. We will refer to these maps as isotopy equivalences. Since $f\colon \mathcal{M} \to I$ is trivial over the open interval, it is clear that isotopy equivalences of open disks will become equivalences under f^{-1}. It will thus suffice to show that if $[-1,a)$ and $[-1,b)$ are half disks and $a < b$, then the inclusion $[-1,a) \hookrightarrow [-1,b)$ over I becomes an equivalence after applying f^{-1}. Pick an isotopy

$$H\colon [-1,1) \times \Delta^1_{\mathrm{ex}} \to I$$

from $[-1,a) \hookrightarrow I$ to $[-1,b) \hookrightarrow I$. We can choose H such that for small enough $-1 < \epsilon$, the restriction $H_\epsilon\colon [-1,\epsilon) \times \Delta^1_{\mathrm{ex}} \to I$ is constantly the inclusion $[-1,\epsilon) \hookrightarrow [-1,1]$. Let $\mathcal{M}_L := f^{-1}[-1,1)$. Since $\mathcal{M}_L \hookrightarrow \mathcal{M}$ is an open substack, it will suffice to produce a lift of H to a Δ^1_{ex}-fiberwise open substack

$$H'\colon \mathcal{M}_L \times \Delta^1_{\mathrm{ex}} \to \mathcal{M}_L$$

implementing an isotopy between the open substacks $f^{-1}[-1,a) \hookrightarrow \mathcal{M}_L$ and $f^{-1}[-1,b) \hookrightarrow \mathcal{M}_L$.

To this end, consider $\mathcal{M}_{L,\epsilon} := f^{-1}[-1,\epsilon)$. By universality of colimits the square

$$\begin{tikzcd}
\mathcal{M}_0 \times (-1,\epsilon) \arrow[r] \arrow[d] & \mathcal{M}_0 \times (-1,1) \arrow[d] \\
\mathcal{M}_{L,\epsilon} \arrow[r] & \mathcal{M}_L
\end{tikzcd}$$

is a pushout in Shv(**Man**), and thus so is the square

$$\begin{tikzcd}
\mathcal{M}_0 \times (-1,\epsilon) \times \Delta^1_{\mathrm{ex}} \arrow[r] \arrow[d] & \mathcal{M}_0 \times (-1,1) \times \Delta^1_{\mathrm{ex}} \arrow[d] \\
\mathcal{M}_{L,\epsilon} \times \Delta^1_{\mathrm{ex}} \arrow[r] & \mathcal{M}_L \times \Delta^1_{\mathrm{ex}}.
\end{tikzcd}$$

The isotopy $H|_{(-1,1)} \colon (-1,1) \times \Delta^1_{\mathrm{ex}} \to I$ lifts to an isotopy

$$H'|_{\mathcal{M}_0 \times (-1,1)} \colon \mathcal{M}_0 \times (-1,1) \times \Delta^1_{\mathrm{ex}} \to \mathcal{M}_L,$$

given by the composition of the top horizontal arrows in the diagram of pullback squares

$$\begin{tikzcd}
\mathcal{M}_0 \times (-1,1) \times \Delta^1_{\mathrm{ex}} \arrow[r] \arrow[d] & \mathcal{M}_0 \times (-1,1) \arrow[r] \arrow[d, "\mathrm{pr}_{(-1,1)}"] & \mathcal{M}_L \arrow[d] \\
(-1,1) \times \Delta^1_{\mathrm{ex}} \arrow[r] & (-1,1) \arrow[r] & [-1,1).
\end{tikzcd}$$

Furthermore, we can define an isotopy

$$H'_{\mathcal{M}_{L,\epsilon}} \colon \mathcal{M}_{L,\epsilon} \times \Delta^1_{\mathrm{ex}} \to \mathcal{M}_L$$

as the top horizontal arrow in the diagram

$$\begin{tikzcd}
\mathcal{M}_{L,\epsilon} \times \Delta^1_{\mathrm{ex}} \arrow[r] \arrow[d] & \mathcal{M}_L \arrow[d] \\
[-1,\epsilon) \times \Delta^1_{\mathrm{ex}} \arrow[r] & [-1,1),
\end{tikzcd}$$

which is a pullback because the bottom arrow is constantly the inclusion $[-1,\epsilon) \hookrightarrow [-1,1)$. By the definition of a pushout, we can glue these isotopies to get a map

$$H' \colon \mathcal{M}_L \times \Delta^1_{\mathrm{ex}} \to \mathcal{M}_L.$$

Again by universality of colimits, the square

$$\begin{tikzcd}
\mathcal{M}_L \times \Delta^1_{\mathrm{ex}} \arrow[r, "H'"] \arrow[d] & \mathcal{M}_L \arrow[d] \\
[-1,1) \times \Delta^1_{\mathrm{ex}} \arrow[r, "H"] & [-1,1)
\end{tikzcd}$$

is a pullback, implying, since H is a Δ^1_{ex}-fiberwise open substack, that H' is. $\square$

The following definition is the counterpart to the construction on topological manifolds given in [AF15, Definition 3.20], that for stratified manifolds given in [AFT17, Definition 8.3.2], and that given in [Hor19, Definition 5.1.1] and [Wee18, Definition 4.20] for G-equivariant manifolds.

Definition 8.4. Let $f \colon \mathcal{M} \to I$ be a collar-gluing of an n-orbifold $\mathcal{M}$. We define the ∞-category $\mathcal{OD}\mathbf{isk}_f$ as the limit of the diagram among ∞-categories

$$\begin{tikzcd}
\mathcal{OD}\mathbf{isk}_{n/\mathcal{M}} \arrow[dr] & & \mathrm{Ar}(\mathcal{O}\mathbf{rb}_{n/\mathcal{M}}) \arrow[dl, "\mathrm{ev}_0"] \arrow[dr, "\mathrm{ev}_1"] & & \mathcal{D}\mathbf{isk}^{\partial}_{1/I} \arrow[dl, "f^{-1}"] \\
& \mathcal{O}\mathbf{rb}_{n/\mathcal{M}} & & \mathcal{O}\mathbf{rb}_{n/\mathcal{M}} &
\end{tikzcd}$$

where $\mathrm{Ar}(\mathcal{O}\mathbf{rb}_{n/\mathcal{M}})$ is the ∞-category of functors $[1] \to \mathcal{O}\mathbf{rb}_{n/\mathcal{M}}$.

The next proposition is the central technical ingredient in our proof of excision. The corresponding result for smooth manifolds, given in the generality of stratified spaces, is [AFT16, Lemma 2.27]. For G-equivariant factorization homology, this is [Hor19, Lemma 5.2.7].

Proposition 8.5. *Let $f\colon \mathcal{M} \to I$ be a collar-gluing of an n-orbifold $\mathcal{M}$. The functor $\mathrm{ev}_0\colon \mathcal{O}\mathcal{D}\mathbf{isk}_f \to \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}}$ is cofinal.*

To prove Proposition 8.5 we will need a model of the mapping spaces $\mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(\mathcal{U}, \mathcal{M})$ for $\mathcal{U}$ a disjoint union of orbifold disks. We will use a stack of *germs* inspired by a construction in [Hor19, Section 3.8].

Definition 8.6. Let $[\mathbb{R}^n/G]$ be an orbifold n-disk. Put a G-equivariant metric on $\mathbb{R}^n$, and define $B_t(\mathbb{R}^n) = \{x \in \mathbb{R}^n \mid ||x|| < t\}$. By Lemma 2.24, the inclusion $B_t(\mathbb{R}^n) \hookrightarrow \mathbb{R}^n$ induces an open substack $\phi_t\colon [\mathbb{R}^n/G] \to [\mathbb{R}^n/G]$, which we will think of as an '*open subdisk of radius t*'.

Definition 8.7. Let $\mathcal{U} = \sqcup_i[\mathbb{R}^n/G]$ be a finite disjoint union of orbifold n-disks, and let $\mathcal{M}$ be an n-orbifold. The map

$$\phi_t\colon \sqcup_i [\mathbb{R}^n/G] \xrightarrow{\sqcup_i \phi_{i,t}} \sqcup_i[\mathbb{R}^n/G]$$

induces a map $\phi_t^*\colon \mathrm{Emb}(\mathcal{U}, \mathcal{M}) \to \mathrm{Emb}(\mathcal{U}, \mathcal{M})$. We define the stack $\mathrm{Germ}(\mathcal{U}, \mathcal{M})$ of *$\mathcal{U}$-germs of* $\mathcal{M}$ to be the colimit of the diagram of stacks

$$\mathrm{Emb}(\mathcal{U}, \mathcal{M}) \xrightarrow{\phi^*_{\frac{1}{2}}} \mathrm{Emb}(\mathcal{U}, \mathcal{M}) \xrightarrow{\phi^*_{\frac{1}{4}}} \mathrm{Emb}(\mathcal{U}, \mathcal{M}) \to \dots$$

indexed by $\mathbb{N}$. We will denote the pointwise colimit taken in *pre-stacks* by $\mathrm{Germ}^{pre}(\mathcal{U}, \mathcal{M})$.

Proposition 8.8. *The map of stacks* $\mathrm{Emb}(\mathcal{U}, \mathcal{M}) \to \mathrm{Germ}(\mathcal{U}, \mathcal{M})$ *induces an equivalence of spaces*

$$\Pi_\infty \mathrm{Emb}(\mathcal{U}, \mathcal{M}) \to \Pi_\infty \mathrm{Germ}(\mathcal{U}, \mathcal{M})$$

between the shapes.

Proof. The map $\phi_t : \sqcup_i[\mathbb{R}^n/G] \to \sqcup_i[\mathbb{R}^n/G] \in \mathcal{O}\mathbf{rb}_n$ is equivalent to the identity $\mathrm{id}_{[\mathbb{R}^n/G]} \in \mathcal{O}\mathbf{rb}_n$, and the shape functor Π_∞ commutes with colimits, so $\Pi_\infty \mathrm{Germ}(\mathcal{U}, \mathcal{M})$ is a colimit of a diagram of equivalences. □

Proof of Proposition 8.5. The functor ev_0 is a Cartesian fibration by [Lur08, Corollary 2.4.7.12] and the fact that pullbacks of Cartesian fibrations are Cartesian. By [Lur08, Lemma 4.1.3.2], we can therefore check cofinality by checking that for each $(\mathcal{U} \hookrightarrow \mathcal{M}) \in \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}}$, for $\mathcal{U} = \sqcup_i[\mathbb{R}^n/G_i]$, the fiber category $\mathrm{ev}_0^{-1}\mathcal{U}$ has contractible classifying space.

Noting that $\mathrm{ev}_0^{-1}\mathcal{U}$ is the comma category $(\mathcal{D}\mathbf{isk}^\partial_{1/I})^{\mathcal{U}/}$, we have an identification of spaces

$$B(\mathrm{ev}_0^{-1}\mathcal{U}) \simeq \operatorname*{colim}_{(V \hookrightarrow I) \in \mathcal{D}\mathbf{isk}^\partial_{1/I}} \mathrm{Map}_{\mathcal{O}\mathbf{rb}_{n/\mathcal{M}}}(\mathcal{U} \hookrightarrow \mathcal{M}, f^{-1}V \hookrightarrow \mathcal{M}).$$

The mapping space in the slice is obtained from the homotopy fiber sequence

$$\mathrm{Map}_{\mathcal{O}\mathbf{rb}_{n/\mathcal{M}}}(\mathcal{U} \hookrightarrow \mathcal{M}, f^{-1}V \hookrightarrow \mathcal{M}) \to \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(\mathcal{U}, f^{-1}V) \to \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(\mathcal{U}, \mathcal{M})$$

for each orbifold disk embedding $\mathcal{U} \hookrightarrow \mathcal{M}$. By universality of colimits, it will thus suffice to show that the map from the colimit

$$\operatorname*{colim}_{(V\hookrightarrow I)\in\mathcal{D}\mathbf{isk}^{\partial}_{1/I}} \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(\mathcal{U}, f^{-1}V) \to \mathrm{Map}_{\mathcal{O}\mathbf{rb}_n}(\mathcal{U}, \mathcal{M})$$

is an equivalence of spaces. By cofinality of the functor $\mathbf{Disk}^{\partial,oe}_{1/I} \to \mathcal{D}\mathbf{isk}^{\partial}_{1/I}$ (Remark 6.20), we can index the colimit diagram by $\mathbf{Disk}^{\partial,oe}_{1/I}$.

By Proposition 8.8 and since Π_∞ preserves colimits, it will suffice to show that the map out of the colimit

$$\operatorname*{colim}_{(V\hookrightarrow I)\in\mathbf{Disk}^{\partial,oe}_{1/I}} \mathrm{Germ}(\mathcal{U}, f^{-1}V) \to \mathrm{Germ}(\mathcal{U}, \mathcal{M})$$

is an equivalence in $\mathrm{Shv}(\mathbf{Man})$. We make the following observations. Firstly, for any $V, W \hookrightarrow I \in \mathbf{Disk}^{\partial,oe}_{1/I}$, we have an equivalence

$$\mathrm{Germ}(\mathcal{U}, f^{-1}V) \times_{\mathrm{Germ}(\mathcal{U},\mathcal{M})} \mathrm{Germ}(\mathcal{U}, f^{-1}W) \simeq \mathrm{Germ}(\mathcal{U}, f^{-1}(V \cap W)).$$

This follows from the equivalence

$$\mathrm{Emb}(\mathcal{U}, f^{-1}V) \times_{\mathrm{Emb}(\mathcal{U},\mathcal{M})} \mathrm{Emb}(\mathcal{U}, f^{-1}W) \simeq \mathrm{Emb}(\mathcal{U}, f^{-1}(V \cap W)),$$

together with the fact that colimits are universal in spaces and that colimits of stacks can be computed by taking colimits in prestacks and stackifying. Secondly, we claim that the map

$$\coprod_{(V\hookrightarrow I)\in\mathbf{Disk}^{\partial,oe}_{1/I}} \mathrm{Germ}(\mathcal{U}, f^{-1}V) \xrightarrow{\rho} \mathrm{Germ}(\mathcal{U}, \mathcal{M})$$

is an effective epimorphism.

Proof that ρ is an effective epimorphism. By [Lur08, Proposition 7.2.1.14], it suffices to check this on the 0-truncated sheaves of sets. Since 0-truncation commutes with sheafification ([Lur08, Proposition 5.5.6.28]), and a local epimorphism of presheaves yields an effective epimorphism of sheaves, it suffices to show that the map of **Set**-valued presheaves

$$\rho^{pre}_{\tau_{\leq 0}} \colon \tau_{\leq 0}(\coprod_{(V\hookrightarrow I)} \mathrm{Germ}^{pre}(\mathcal{U}, f^{-1}V)) \to \tau_{\leq 0}(\mathrm{Germ}^{pre}(\mathcal{U}, \mathcal{M}))$$

is a local epimorphism. Since truncation of presheaves is applied objectwise, this means showing that for each $T \in \mathbf{Man}$ and each section $y \in \tau_{\leq 0}(\mathrm{Germ}^{pre}(\mathcal{U}, \mathcal{M})(T))$, there is an open cover $\{T_t\}$ of T and sections

$$x_t \in \tau_{\leq 0}(\coprod_V \mathrm{Germ}^{pre}(\mathcal{U}, f^{-1}V)(T_t))$$

such that $\rho^{pre}_{\tau_{\leq 0}}(T_t)(x_t) = y|_{T_t}$. Pick a representative $y \colon \mathcal{U} \times T \to \mathcal{M}$. This is really a representative of a representative: first choose a representative of the equivalence class of $y \in \pi_0(\mathrm{Germ}^{pre}(\mathcal{U}, \mathcal{M})(T))$, and then choose a representative $y \in \mathrm{Emb}(\mathcal{U}, \mathcal{M})(T)$ in the colimit; we denote all of these by y by abuse of notation. Now define $W_V := (\mathcal{U} \times T) \times_{\mathcal{M}} f^{-1}V$ for each $V \hookrightarrow I$. For $\sqcup_i \mathbb{R}^n \times T \to \mathcal{U} \times T$ an atlas, define $\widetilde{W}_V := W_V \times_{\mathcal{U}\times T} (\sqcup_i \mathbb{R}^n \times T)$. This is just an open of $\sqcup_i \mathbb{R}^n \times T$, since it is a pullback of an open substack. For each $t \in T$, consider the set of zeros $\{(0_i, t) \in \sqcup_i \mathbb{R}^n \times T\}$. The map of topological spaces $\sqcup_i \mathbb{R}^n \times T \to I$ yields a finite number of points in I, so we can choose a finite disjoint union of disks $V_t \hookrightarrow I$

containing all of these points. In particular, $\widetilde{W}_{V_t}$ will contain the zeros $\{(0_i, t)\}$. For each i, pick a G_i-equivariant open embedding $\phi^i_t\colon \mathbb{R}^n \to \mathbb{R}^n$ and an open T^i_t such that $(0_i, t) \in \phi^i_t(\mathbb{R}^n) \times T^i_t \subset \widetilde{W}_{V_t}$. Letting $T_t := \cap_i T^i_t$, we see that the open $(\sqcup_i \phi^i_t(\mathbb{R}^n)) \times T_t \subset \widetilde{W}_{V_t}$ contains the zeros $\{(0_i, t)\}$. Since we chose the open embeddings ϕ^i_t to be G_i-equivariant, we get an induced map $\sqcup_i[\mathbb{R}^n/G_i] \times T_t \to \mathcal{M}$ which by construction factors into

$$\sqcup_i[\mathbb{R}^n/G_i] \times T_t \xrightarrow{x_t} f^{-1}(V_t) \to \mathcal{M},$$

where $x_t \in \mathrm{Germ}^{pre}(\mathcal{U}, f^{-1}V_t)(T_t)$. Now notice that the map $\sqcup_i[\mathbb{R}^n/G_i] \times T_t \to \mathcal{M}$ is equivalent to $y|_{T_t} \in \mathrm{Germ}^{pre}(\mathcal{U}, \mathcal{M})(T_t)$, so we have proven the claim. $\square$

To conclude the proof of the proposition, note that equivalences in an ∞-topos can be checked after pulling back along an effective epimorphism on the target. Considering the pullback square

$$\begin{tikzcd}
P \arrow[r] \arrow[d, two heads] & \coprod_{(W\hookrightarrow I)\in \mathbf{Disk}^{\partial,oe}_{1/I}} \mathrm{Germ}(\mathcal{U}, f^{-1}W) \arrow[d, two heads] \\
\mathrm{colim}_{(V\hookrightarrow I)\in \mathbf{Disk}^{\partial,oe}_{1/I}} \mathrm{Germ}(\mathcal{U}, f^{-1}V) \arrow[r] & \mathrm{Germ}(\mathcal{U}, \mathcal{M}),
\end{tikzcd}$$

universality of colimits and our first claim give that

$$P \simeq \coprod_{(W\hookrightarrow I)\in \mathbf{Disk}^{\partial,oe}_{1/I}} \ \underset{(V\hookrightarrow I)\in \mathbf{Disk}^{\partial,oe}_{1/I}}{\mathrm{colim}} \mathrm{Germ}(\mathcal{U}, f^{-1}(V\cap W)).$$

Denote by $(\mathbf{Disk}^{\partial,oe}_{1/I})_{\subset W}$ the subposet of disjoint unions of disks contained in W. Since each $V \cap W$ is a disjoint union of disks in I, we have an equivalence

$$\underset{(V\hookrightarrow I)\in \mathbf{Disk}^{\partial,oe}_{1/I}}{\mathrm{colim}} \mathrm{Germ}(\mathcal{U}, f^{-1}(V\cap W)) \simeq \underset{(V\hookrightarrow I)\in (\mathbf{Disk}^{\partial,oe}_{1/I})_{\subset W}}{\mathrm{colim}} \mathrm{Germ}(\mathcal{U}, f^{-1}V).$$

Since $W \hookrightarrow I$ is final in $(\mathbf{Disk}^{\partial,oe}_{1/I})_{\subset W}$, this colimit is simply $\mathrm{Germ}(\mathcal{U}, f^{-1}W)$, implying that

$$P \simeq \coprod_{(W\hookrightarrow I)\in \mathbf{Disk}^{\partial}_{1/I}} \mathrm{Germ}(\mathcal{U}, f^{-1}W),$$

concluding the proof of the proposition. $\square$

Definition 8.9. Let $f\colon \mathcal{M} \to I$ be a collar-gluing of an n-orbifold $\mathcal{M}$ and let $A\colon \mathcal{OD}\mathbf{isk}_n \to \mathcal{V}^{\otimes}$ be a symmetric monoidal functor to a $\otimes$-presentable symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$. We will call the composite functor

$$f_*A\colon \mathcal{D}\mathbf{isk}^{\partial}_{1/I} \xrightarrow{f^{-1}} \mathcal{O}\mathbf{rb}_{n/\mathcal{M}} \xrightarrow{\int_{-} A} \mathcal{V}$$

the *pushforward* of A along f.

The following proposition is the orbifold-counterpart to [AFT16, Theorem 2.25] for stratified spaces, which is [AF15, Proposition 3.23] for topological manifolds.

Proposition 8.10. *Let $f\colon \mathcal{M} \to I$ be a collar-gluing of an n-orbifold $\mathcal{M}$ and let $A\colon \mathcal{OD}\mathbf{isk}_n \to \mathcal{V}^{\otimes}$ be a symmetric monoidal functor to a $\otimes$-presentable symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$. There is an equivalence*

$$\mathrm{colim}(f_*A\colon \mathcal{D}\mathbf{isk}^{\partial}_{1/I} \xrightarrow{f^{-1}} \mathcal{O}\mathbf{rb}_{n/\mathcal{M}} \xrightarrow{\int_{-} A} \mathcal{V}) \xrightarrow{\simeq} \int_{\mathcal{M}} A.$$

Proof. The proof given for stratified spaces and in [AFT16, Theorem 2.25], also explained in the context of topological manifolds in [AF15, Proposition 3.23], applies directly in our setting of orbifolds. For the reader's convenience, we explain the proof here. By definition, the colimit on the left hand side is the double colimit

$$\operatorname{colim}(f_*A) \simeq \operatorname*{colim}_{V\in\mathcal{D}\mathbf{isk}^{\partial}_{1/I}} \operatorname*{colim}_{U\in\mathcal{O}\mathcal{D}\mathbf{isk}_{n/f^{-1}V}} A(U).$$

We claim we have an equivalence

$$\operatorname*{colim}_{V\in\mathcal{D}\mathbf{isk}^{\partial}_{1/I}} \operatorname*{colim}_{U\in\mathcal{O}\mathcal{D}\mathbf{isk}_{n/f^{-1}V}} A(U) \simeq \operatorname*{colim}_{(V,U)\in\mathcal{D}\mathbf{isk}_f} A(U).$$

To see why this is true, consider the left Kan extension

$$\begin{tikzcd} \mathcal{D}\mathbf{isk}_f \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] & \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}} \arrow[r] & \mathcal{O}\mathcal{D}\mathbf{isk}_n \arrow[r, "A"] & \mathcal{V} \\ \mathcal{D}\mathbf{isk}^{\partial}_{1/I}. \arrow[urrr, dashed, "\mathrm{Lan}"'] & & & \end{tikzcd}$$

The map $\mathrm{ev}_1\colon \mathcal{D}\mathbf{isk}_f \to \mathcal{D}\mathbf{isk}^{\partial}_{1/I}$ is a coCartesian fibration, so that for each $V \in \mathcal{D}\mathbf{isk}^{\partial}_{1/I}$, the functor

$$\mathrm{ev}_1^{-1}(V) \to (\mathcal{D}\mathbf{isk}_f)_{/V}$$

is cofinal. Since we can therefore compute $\mathrm{Lan}(V)$ as the colimit over the fiber $\mathrm{ev}_1^{-1}(V)$, we have

$$\mathrm{Lan}(V) \simeq \operatorname*{colim}_{U\in \mathrm{ev}_1^{-1}(V)} A(U) \simeq \operatorname*{colim}_{U\in\mathcal{O}\mathcal{D}\mathbf{isk}_{n/f^{-1}V}} A(U)$$

Now by [Lur08, Proposition 4.3.3.7], the colimit of Lan agrees with the colimit of the composite $\mathcal{D}\mathbf{isk}_f \to \mathcal{V}$. So we have equivalences

$$\operatorname*{colim}_{(V,U)\in\mathcal{D}\mathbf{isk}_f} A(U) \simeq \operatorname*{colim}_{V\in\mathcal{D}\mathbf{isk}^{\partial}_{1/I}} \mathrm{Lan}(V) \simeq \operatorname*{colim}_{V\in\mathcal{D}\mathbf{isk}^{\partial}_{1/I}} \operatorname*{colim}_{U\in\mathcal{O}\mathcal{D}\mathbf{isk}_{n/f^{-1}V}} A(U),$$

as claimed. By Proposition 8.5 the functor $\mathrm{ev}_0\colon \mathcal{D}\mathbf{isk}_f \to \mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}}$ is cofinal, and we thus have

$$\operatorname*{colim}_{(V,U)\in\mathcal{D}\mathbf{isk}_f} A(U) \simeq \operatorname*{colim}_{U\in\mathcal{O}\mathcal{D}\mathbf{isk}_{n/\mathcal{M}}} A(U) \simeq \int_{\mathcal{M}} A,$$

concluding the proof. $\square$

Lemma 8.11. *There is a functor $\Delta^{\mathrm{op}} \to \mathcal{D}\mathbf{isk}^{\partial}_{1/I}$ which is cofinal. On objects, the functor sends $[n] \in \Delta^{\mathrm{op}}$ to an embedding*

$$[-1,0) \sqcup \mathbb{R}^{\sqcup n} \sqcup (0,1] \hookrightarrow [-1,1]$$

Proof. This is stated in the proof of [AFT16, Proposition 2.34]. $\square$

Given a collar-gluing of an n-orbifold $f\colon \mathcal{M} \to I$ and an algebra $A\colon \mathcal{O}\mathcal{D}\mathbf{isk}_n \to \mathcal{V}^{\otimes}$ to a $\otimes$-presentable symmetric monoidal ∞-category $\mathcal{V}$, the colimit of the composite map

$$\Delta^{\mathrm{op}} \to \mathcal{D}\mathbf{isk}^{\partial}_{1/I} \xrightarrow{f^{-1}} \mathcal{O}\mathbf{rb}_{n/\mathcal{M}} \xrightarrow{\int_{-} A} \mathcal{V}$$

recovers factorization homology.

Denote by $\mathcal{M}_-$ the suborbifold $f^{-1}([-1,1))$ and by $\mathcal{M}_+$ the suborbifold $f^{-1}((-1,1])$. Writing out this simplicial object, we obtain

$$\cdots \substack{\to\\\to\\\to} \int_{\mathcal{M}_-} A \quad \bigotimes_{\mathcal{V}} \quad \int_{\mathcal{M}_0\times\mathbb{R}} A \quad \bigotimes_{\mathcal{V}} \quad \int_{\mathcal{M}_+} A \quad \rightrightarrows \quad \int_{\mathcal{M}_-} A \quad \bigotimes_{\mathcal{V}} \quad \int_{\mathcal{M}_+} A,$$

with $[n] \in \Delta^{\mathrm{op}}$ mapping to

$$\int_{\mathcal{M}_-} A \quad \bigotimes_{\mathcal{V}} \quad (\int_{\mathcal{M}_0\times\mathbb{R}} A)^{\otimes n} \quad \bigotimes_{\mathcal{V}} \quad \int_{\mathcal{M}_+} A.$$

Following [AFT16, Section 2] and [AF15, Section 3], we will call this simplicial object a *two-sided bar construction* and its colimit

$$\int_{\mathcal{M}_-} A \bigotimes_{\int_{\mathcal{M}_0\times\mathbb{R}}} \int_{\mathcal{M}_+} A \simeq \int_{\mathcal{M}} A$$

a *relative tensor product*. We note that neither paper justifies that this object is a relative tensor product in the sense of [Lur, 4.4.2].

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