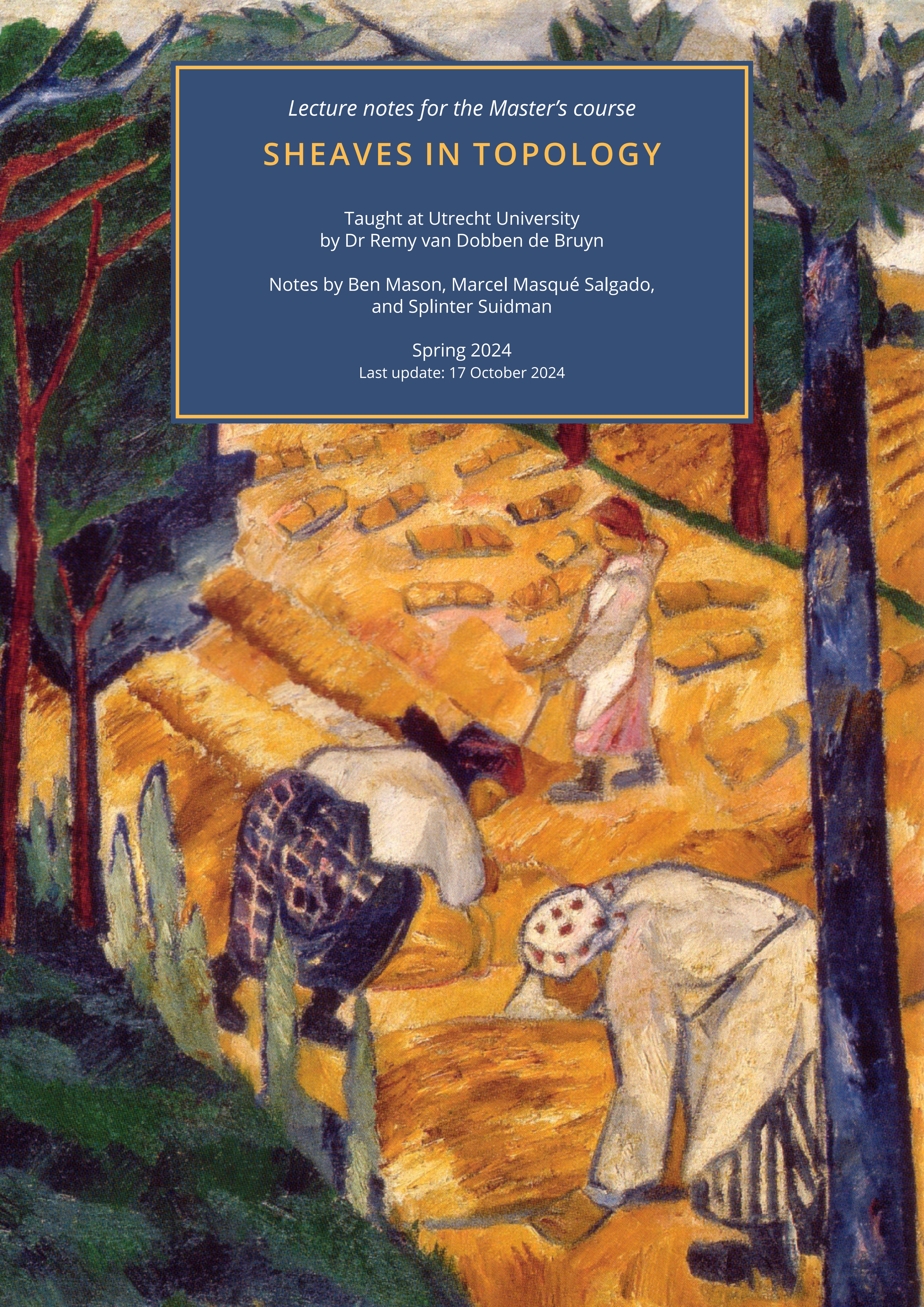

Lecture notes for the Master's course

SHEAVES IN TOPOLOGY

Taught at Utrecht University
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Contents

Introduction

These notes are based on lectures given by Remy van Dobben de Bruyn for the Master's course *Sheaves in Topology*, taught at Utrecht University in the spring semester of 2023–2024.[1]

The prerequisites for this course are a solid understanding of point-set topology, basic knowledge of fundamental groups and covering spaces, familiarity with the language of categories, and a working knowledge of modules over rings.

Recommended literature

Standard works:

- Iversen, *Cohomology of sheaves* [Ive86]
- Bredon, *Sheaf theory* [Bre97]
- Tennison, *Sheaf theory* [Ten75]
- Kashiwara and Schapira, *Sheaves on manifolds* [KS94]
- Stacks project authors, *The Stacks project* [Sta24, Chapter 006A] (chapter on sheaves)

More advanced texts:

- Dimca, *Sheaves in topology* [Dim04]
- Mac Lane and Moerdijk, *Sheaves in geometry and logic* [MM94]

Exodromy correspondence (research papers):

- Treumann, 'Exit paths and constructible stacks' [Tre09]
- Curry and Patel, 'Classification of constructible cosheaves' [CP20]

Course content

The first four lectures introduce presheaves and sheaves on a topological space X and describe an equivalence of categories between local homeomorphisms over X and sheaves on X. For the special case of locally constant sheaves there is an equivalence to the category of covering spaces of X.

Some categorical properties of sheaves, and constructions such as the pushforward and the pullback, are discussed next. After an introduction to homological algebra (which is independent of the content on sheaves), the notion of sheaf cohomology is treated. This takes up Lecture 8, Lecture 9, Lecture 10, Lecture 11, and Lecture 12.

The real fun begins when the homological algebra is applied to sheaves of abelian groups. One of the main results of the course, the proper base change theorem, is proven in Lecture 15

[1] See https://cursusplanner.uu.nl/course/WISM501/2023/SEM2 for the course description.

for paracompact Hausdroff and locally compact Hausdorff spaces. The treatment of sheaf cohomology ends with a discussion of Čech cohomology.

The last three weeks (Lecture 18, Lecture 19, Lecture 20) are reserved entirely to having fun, and as such were not examinable material in the 2024 version of the course.

write after we write Lecture 19.

course content

LECTURE 1

Motivation, sheaves and presheaves

1.1 Sheaves and presheaves

Definition 1.1.1 · Let X be a topological space. Write $\mathrm{Open}(X)$ for the partially ordered set of opens of X. A *presheaf* of sets on X is a functor $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$.

By changing the codomain we can obtain, for example, presheaves of abelian groups. In this course, we will focus almost entirely on presheaves of sets and of abelian groups. Let $U \subseteq V$ be open sets of X. The inclusion under F gives a *restriction* map $F(V) \to F(U)$. The naming comes from the following example:

Example 1.1.2 · Let X and Z be topological spaces. The assignment $h_Z : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ by $U \mapsto \mathbf{Top}(U, Z)$ can be turned into a presheaf: given $U \subseteq V$ opens of X, define

$$r_{UV} : \mathbf{Top}(V, Z) \to \mathbf{Top}(U, Z)$$

by $f \mapsto f|_U$.

We generalise the notation of function restriction. For $F(V) \to F(U)$, we denote the map pointwise by $s \mapsto s|_U$.

Definition 1.1.3 · Let X be a topological space and $\mathcal{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ a presheaf on X. We call $\mathcal{F}$ a *sheaf* if it satisfies the *sheaf condition*, i.e., if for every open $U \subseteq X$ and every open cover $(U_i)_{i\in I}$ of U with $\bigcup_i U_i = U$, the map

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i), \quad s \mapsto (s|_{U_i})_{i\in I}$$

(i) is injective, and

(ii) its image satisfies a *gluing condition*: it is given by $\{ (s_i)_{i\in I} \mid s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \forall i, j \in I \}$.

Remark 1.1.4 · One checks that the sheaf condition is equivalent to requiring that

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equaliser diagram for all U open in X and for all $(U_i)_{i\in I}$ open covers of U, where α : $(s_i)_{i\in I} \mapsto s_i|_{U_i\cap U_j}$ and $\beta : (s_i)_{i\in I} \mapsto s_j|_{U_i\cap U_j}$.

Lemma 1.1.5 · *The presheaf h_Z from Example 1.1.2 is a sheaf.*

Proof. If two functions agree on every open of a cover of U they agree on U, this gives Definition 1.1.3(i). For (ii), we use the pasting lemma. □

Example 1.1.6 · Let Z be a discrete topological space, let X be a topological space. Given an open subset U of X, a map $f : U \to Z$ is continuous if and only if it is locally constant. The sheaf h_Z is called the *constant sheaf* on the set Z, labelled $\underline{Z}$ or $\underline{Z}_X$. Explicitly, $\underline{Z}$ is given by

$$\underline{Z} : U \mapsto \mathbf{Top}(U, Z),$$

where Z is endowed with the discrete topology.

Example 1.1.7 · If X is a manifold, then the assignment $U \mapsto C^\infty(U, \mathbb{R})$ is a sheaf of $\mathbb{R}$ vector spaces. One can show that the assignment $U \mapsto \Omega^k(U)$ (smooth differential k-forms) is a sheaf.

Lemma 1.1.8 (sheaf of sections) · *Let $f : Y \to X$ be a continuous map of topological spaces. The assignment on opens of X given by*

$$h_{Y/X} : U \mapsto \{ s : U \to f^{-1}(U) \mid f \circ s = \mathrm{id}_U \} =: \mathbf{Top}_{/X}(U, Y)$$

is a sheaf.

Proof. One can prove the above lemma in a similar way we proved Lemma 1.1.5. Alternatively, consider the diagram

$$\begin{array}{ccc} \mathbf{Top}_{/X}(U, Y) & \longrightarrow & \mathbf{Top}(U, Y) \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f\circ -} \\ \{U \to X\} & \longrightarrow & \mathbf{Top}(U, X) \end{array}$$

and check that it is a pullback. We will come back to this in more detail in later lectures. □

The sheaf $h_{Y/X}$ of the lemma is called the *sheaf of sections*. The example in the lemma above is why the elements of $\mathcal{F}(U)$ for an arbitrary sheaf $\mathcal{F}$ are more generally also called *sections*.

Example 1.1.9 · Let $f : Y \to X$ be the two-to-one cover of the circle: $f : z \mapsto z^2$, with $X := S^1$ and $Y := S^1$. On small intervals $U \subseteq X$ we get $f^{-1}(U) \cong U \times \{1, 2\}$. We thus have two sections: $U \mapsto (U, 1)$ and $U \mapsto (U, 2)$, so $h_{Y/X}(U)$ has two elements. On V a union of small intervals, we get $2^{|\pi_0(V)|}$ elements, where $|\pi_0(V)|$ is the number of path components of V. On $W = X$, we get no sections. If $s : X \to Y$ is a section, then the induced map $s_* : \pi_1(X) \to \pi_1(Y)$ is a section to the map $f_* : \pi_1(Y) \to \pi_1(X)$. But this induced map is multiplication by 2, and it does not have a section.

LECTURE 2

Morphisms of (pre)sheaves, Yoneda lemma, étalé space

'You can do what you want, it's a free world (on one generator).'

2.1 Morphisms of (pre)sheaves

Definition 2.1.1 · Let $\mathscr{C}$ be a small category. Then $\mathbf{PSh}(\mathscr{C})$ is the functor category $\mathbf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$: its objects are functors $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$, and its morphisms $\alpha : F \Rightarrow G$ are *natural transformations*, that is, collections of functions $\alpha_X : F(X) \to G(X)$ for all $X \in \mathscr{C}$ such that the diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{\alpha_Y} & G(Y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(X) & \xrightarrow[\alpha_X]{} & G(X) \end{array}$$

commutes for all $f : X \to Y$ in $\mathscr{C}$.

Definition 2.1.2 · Let X be a topological space. Then the category of presheaves on X is $\mathbf{PSh}(X) := \mathbf{PSh}(\mathrm{Open}(X))$ and the category of sheaves on X is the full subcategory $\mathbf{Sh}(X) \subseteq \mathbf{PSh}(X)$ on the sheaves.

Lemma 2.1.3 · *Let $\mathscr{C}$ be a small category. Then:*

(i) $\mathbf{PSh}(\mathscr{C})$ *is a category.*

(ii) The category $\mathbf{PSh}(\mathscr{C})$ *has all (small) limits and colimits, and they are computed objectwise; e.g., for presheaves $F, G \in$* $\mathbf{PSh}(\mathscr{C})$ *the natural map*

$$(F \times G)(U) \to F(U) \times G(U)$$

is an isomorphism for all open $U \subseteq X$.

(iii) A natural transformation $\alpha : F \Rightarrow G$ between presheaves F and G on $\mathscr{C}$ is invertible if and only if the component $\alpha_X : F(X) \to G(X)$ is a bijection for all $X \in \mathscr{C}$.

Remark 2.1.4 · Colimits in the category $\mathbf{Sh}(X)$ of sheaves will be more complicated.

Example 2.1.5 · Recall that we defined a sheaf h_Z on X for $Z \in \mathbf{Top}$ (Example 1.1.2). If $g : Z \to Z'$ is a continuous map, then we get a natural transformation $h_Z \Rightarrow h_{Z'}$ with component

$$\mathbf{Top}(U, Z) \to \mathbf{Top}(U, Z'), \quad f \mapsto g \circ f$$

at $U \in \mathrm{Open}(X)$. One checks that this is natural.

Example 2.1.6 · For the sheaf of sections (Lemma 1.1.8), a map $g : Y \to Y'$ over X induces a natural transformation $h_{Y/X} \Rightarrow h_{Y'/X}$ with component

$$\mathbf{Top}_{/X}(U, Y) \to \mathbf{Top}_{/X}(U, Y'), \quad s \mapsto g \circ s$$

at $U \in \mathrm{Open}(X)$. This is again natural in U.

In fact, the sheaf h_Z is a special case of $h_{Y/X}$:

Lemma 2.1.7 · *If* $Y = Z \times X \xrightarrow{\mathrm{pr}_X} X$ *in* $\mathbf{Top}_{/X}$, *then the sheaves* $h_{Y/X}$ *and* h_Z *are isomorphic.* Picture

Proof. For $U \in \mathrm{Open}(X)$, define

$$\mathbf{Top}_{/X}(U, Z \times X) \to \mathbf{Top}(U, Z), \quad s \mapsto \mathrm{pr}_Z \circ s$$

and

$$\mathbf{Top}(U, Z) \to \mathbf{Top}_{/X}(U, Z \times X), \quad f \mapsto (u \mapsto (f(u), u)).$$

These maps are inverses. Both transformations are natural. For the first: given an inclusion of opens $U \subseteq V \subseteq X$, the diagram

$$\begin{array}{ccc} \mathbf{Top}_{/X}(V, Z \times X) & \xrightarrow{\mathrm{pr}_Z \circ -} & \mathbf{Top}(V, Z) \\ {\scriptstyle (-)|_U}\downarrow & & \downarrow{\scriptstyle (-)|_U} \\ \mathbf{Top}_{/X}(U, Z \times X) & \xrightarrow[\mathrm{pr}_Z \circ -]{} & \mathbf{Top}(U, Z) \end{array}$$

commutes. □

Example 2.1.8 · Let $Y \to X$ be the two-to-one cover $S^1 \to S^1$. Then $h_{Y/X}$ is not isomorphic to h_Z for any $Z \in \mathbf{Top}$ (but it is locally isomorphic to $h_{\{1,2\}}$, as we will see later). Ref back

Suppose $h_{Y/X} \cong h_Z$ for $Z \in \mathbf{Top}$. Then $Z \neq \varnothing$, for

$$h_\varnothing(U) = \begin{cases} \varnothing & \text{if } U \neq \varnothing, \\ * & \text{if } U = \varnothing. \end{cases}$$

But we have seen that $h_{Y/X}$ is nonempty for a small enough interval. So $Z \neq \varnothing$, so constant maps show that $h_Z(U) \neq \varnothing$ for all U. But last lecture, we saw that $h_{Y/X}(X) = \varnothing$. Ref

2.2 Yoneda lemma

Definition 2.2.1 · Let $\mathscr{C}$ be a small category. Then the *representable presheaf* on $X \in \mathscr{C}$ is the presheaf

$$h_X : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}, \quad Y \mapsto \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

with restriction map

$$f^* : \mathrm{Hom}_{\mathscr{C}}(X, Y') \to \mathrm{Hom}_{\mathscr{C}}(X, Y), \quad g \mapsto g \circ f.$$

induced by $f : Y \to Y'$ in $\mathscr{C}$.

Remark 2.2.2 · The sheaf h_Z on X from Example 1.1.2 is *not* a representable presheaf on $\mathrm{Open}(X)$. The sheaf h_Z sends an open $U \subseteq X$ to the set $\mathbf{Top}(U, Z)$ of all continuous maps $U \to Z$, whereas the representable presheaf h_V represented by an open $V \subseteq X$ sends $U \subseteq X$ to the set $\mathrm{Hom}_{\mathrm{Open}(X)}(U, V)$ of inclusions $U \hookrightarrow V$ (a subsingleton set). The sheaf h_Z can be regarded as the *restriction* of the representable presheaf $h_Z = \mathrm{Hom}_{\mathbf{Top}}(-, Z)$ on the (non-small) category of spaces to the (non-full) subcategory $\mathrm{Open}(X) \subseteq \mathbf{Top}$.

In some sense, the representable presheaf represented by X is 'freely generated' by the section $\mathrm{id}_X \in h_X(X)$.

Lemma 2.2.3 (Yoneda lemma [Rie16, Theorem 2.2.4]) · *Let $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor, $X \in \mathscr{C}$. Then the map*

$$\Phi : \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(h_X, F) \to F(X), \quad \alpha \mapsto \alpha_X(\mathrm{id}_X)$$

is a bijection that is natural in F and X.

Proof. We leave the naturality in F and X as an exercise. Part of the exercise is to work out what is meant by naturality in F and X. The inverse of Φ will be a map

maybe add details

$$\Psi : F(X) \to \mathrm{Hom}_{\mathbf{PSh}(C)}(h_X, F)$$

defined by $s \mapsto (f \mapsto F(f)(s))_{Y \in \mathscr{C}}$. The first thing to check is that this map lands in $\mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(h_X, F)$. That is, we need to check $\Psi(s)$ is a natural transformation. This is true: given $g : Y \to Z$ in $\mathscr{C}$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}(Z,X) & \xrightarrow{\Psi(s)_Z} & F(Z) \\ {\scriptstyle -\circ g}\downarrow & & \downarrow{\scriptstyle F(g)} \\ \mathrm{Hom}(Y,X) & \xrightarrow[\Psi(s)_Y]{} & F(Y) \end{array}$$

commutes, since we have $F(g)(\Psi(s)_Z(f)) = F(g)(F(f)(s)) = F(fg)(s) = \Psi(s)_Y(fg)$. We check that Ψ provides an inverse for Φ. One side is immediate: $\Phi\Psi(s) = \Psi(s)_X(\mathrm{id}_X) = F(\mathrm{id}_X)(s) = s$. For the converse, let $\alpha : h_X \to F$ be a natural transformation. We check that $\Psi\Phi(\alpha) = \alpha$. Let $Y \in \mathscr{C}$, and $f \in \mathrm{Hom}(Y, X)$. The naturality of α gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(X,X) & \xrightarrow{\alpha_X} & F(X) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}(Y,X) & \xrightarrow[\alpha_Y]{} & F(Y) \end{array}$$

We now have $\Psi(\Phi\alpha)_Y(f) = F(f)(\Phi\alpha) = F(f)(\alpha_X(\mathrm{id}_X)) = \alpha_Y(\mathrm{id}_X \circ f) = \alpha_Y(f)$ by plugging in id_X into the diagram above. □

2.3 Étalé space

In the coming lectures, we will show that every sheaf on a space X is isomorphic to $h_{Y/X}$ for some map $Y \to X$. We will construct a functor $\mathrm{sp} : \mathbf{PSh}(X) \to \mathbf{Top}_{/X}$ which sends a presheaf to its *étalé space* (also called *espace étalé*, (wrongly) *étale space* or *sheaf space* (but we do not like that)).

The idea is as follows: a presheaf F is determined by its values $F(U)$ for opens $U \subseteq X$ and the restriction maps. By the Yoneda lemma, the values can be written as $F(U) \cong \mathrm{Hom}_{\mathbf{PSh}(X)}(h_U, F)$. For h_U, the étalé space will be the space $U \to X$ over X. In general, a presheaf F is 'generated by copies of h_U modulo relations from the restrictions'. We define $\mathrm{sp}(F) \to X$ by glueing copies of U 'along the same colimit diagram' (but now in $\mathbf{Top}_{/X}$ instead of $\mathbf{PSh}(X)$).

Definition 2.3.1 · Let F be a presheaf on X. The *étalé space* $\mathrm{sp}(F)$ is the quotient of the coproduct

$$\coprod_{U \in \mathrm{Open}(X)} F(U) \times U$$

(where $F(U)$ is endowed with the discrete topology), with notation $(s, x)_U$ for the point given by $s \in F(U)$ and $x \in U$ in the factor indexed by U, by the equivalence relation generated by

$$(s|_U, x)_U \sim (s, x)_V$$

for $(s, x) \in F(V) \times U$ for $x \in U \subseteq V$.

Categorically, the étalé space is the coequaliser

$$\operatorname{sp}(F) = \operatorname{coeq}\Big(\coprod_{U \subseteq V \subseteq X} F(V) \times U \rightrightarrows \coprod_{U \subseteq X} F(U) \times U\Big)$$

where the arrows are given by

$$\begin{array}{ccccc} F(U) \times U & \xleftarrow{\operatorname{incl}^* \times \operatorname{id}} & F(V) \times U & \xrightarrow{\operatorname{id} \times \operatorname{incl}} & F(V) \times V \\ \Downarrow & & \Downarrow & & \Downarrow \\ (s|_U, x) & \longleftarrow\!\!\dashv & (s, x) & \longmapsto & (s, x) \end{array}$$

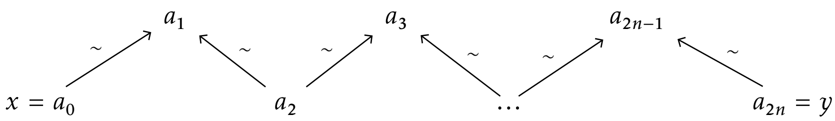
where incl : $U \hookrightarrow V$ denotes the inclusion map.

Alternatively, the étalé space $\operatorname{sp}(F)$ is the coend

$$\operatorname{sp}(F) = \int^{U \in \operatorname{Open}(X)} F(U) \times U.$$

To understand the étalé space a bit better, we have to understand the equivalence relation generated by the given relation, with respect to which we take the quotient.

Remark 2.3.2 · Let $\sim$ be a relation on a set X. The equivalence relation $\approx$ on X generated by $\sim$ can be explicitly defined as follows: we say $x \approx y$ if and only if there is a finite zigzag



of elements $a_i \in X$ such that $x = a_0$, $y = a_{2n}$, $a_{2i} \sim a_{2i+1}$ for $0 \leqslant i < n$ and $a_{2i} \sim a_{2i-1}$ for $0 < i \leqslant n$.

Example 2.3.3 · To compute the coequaliser of the diagram of sets

$$\{1, 2\} \underset{-\cdot 2}{\overset{-\cdot 1}{\rightrightarrows}} \{1, 2, 3, 4\}$$

where the maps multiply by one and two, respectively, we need to understand the equivalence relation generated by $x \sim 2x$ for all $x \in \{1, 2\}$. That is, we have $1 \sim 2$ and $2 \sim 4$, and nothing more. The equivalence classes in the relation $\approx$ generated by $\sim$ then are $\{1, 2, 4\}$ and $\{3\}$.

Lemma 2.3.4 · *If $(s, x)_U \in F(U) \times U$ and $(t, y)_V \in F(V) \times V$, then $(s, x)_U \approx (t, y)_V$ (that is, they are equivalent under the equivalence relation generated by $\sim$ as in Definition 2.3.1) if and only if $x = y$ and there exists an open neighbourhood $W \subseteq U \cap V$ of x such that $s|_W = t|_W$.*

One can prove this lemma using the explicit description of $\approx$ given in Remark 2.3.2, but this is difficult combinatorics! A better proof shows that the relation on the right in the equivalence of the lemma is an equivalence relation and that the former statement implies the latter (i.e., '$\sim$ implies $\approx$').

Example 2.3.5 · If $F = h_U$ is the representable presheaf on X represented by an open $U \subseteq X$ (see Figure 2.1), then

$$F(V) = \operatorname{Hom}_{\operatorname{Open}(X)}(V, U) = \begin{cases} * & \text{if } V \subseteq U, \\ \varnothing & \text{if } V \not\subseteq U. \end{cases}$$

The étalé space $\operatorname{sp}(F)$ is the quotient of

$$\coprod_{V \subseteq X} F(V) \times V = \coprod_{V \subseteq U} V$$

by the equivalence relation generated by $(*, x)_V \sim (*, x)_{V'}$ for all $V, V' \subseteq U$. This just leaves U, so $\operatorname{sp}(h_U) = U$.

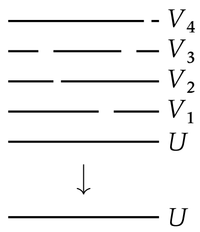


Figure 2.1 · The presheaf $F = h_U$ represented by $U \subseteq X$ of Example 2.3.5.

Remark 2.3.6 · In general, the étalé space $\mathrm{sp}(F)$ is not 'computable'. It is, however, when F is *constructible*, which we will see in one of the final lectures. Ref back

Proposition 2.3.7 · *The construction of the étalé space defines a functor* $\mathbf{Sh}(X) \to \mathbf{Top}_{/X}$.

Proof. We can cheat and use the unproven remark that the étalé space is a coend. □ prove?

LECTURE 3

Local homeomorphisms, sheaf/space adjunction

??

3.1 Local homeomorphisms

> 'I don't like annoying stuff, so let's do it the non-annoying way.'

Definition 3.1.1 · A continuous function $f : Y \to X$ is said to be a *local homeomorphism* if for all $y \in Y$ there exists an open neighbourhood $V_y \subseteq Y$ such that $f|_{V_y} : V_y \to f(V)$ is a homeomorphism onto an open subset of X.

Later we will see that we may equivalently define an étalé space over X as a topological space Y together with a local homeomorphism $Y \to X$.

Remark 3.1.2 · A local homeomorphism $f : Y \to X$ is always open. Namely let $W \subseteq Y$ be open, then it can be covered by opens of the form $W \cap V_y$, where $f|_{V_y} : V_y \to f(V_y)$ are homeomorphisms. Consequently,

$$f(W) = f(\bigcup_{y \in Y} V_y \cap W) = \bigcup_{y \in Y} f(V_y \cap W).$$

Therefore, since $f(V_y \cap W) = f|_{V_y}(V_y \cap W)$ is open in $f(V_y)$, it is open in X, thus $f(W)$ is open.

Lemma 3.1.3 · *Let*

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ & {}_h\searrow \quad \swarrow_f & \\ & X & \end{array}$$

be a commutative triangle in **Top**. *Then,*

(i) If f and g are local homeomorphisms then so is h.

(ii) If f and h are local homeomorphisms then so is g.

(iii) If g is open and surjective and h is a local homeomorphism then so are f and g.

Proof. Argue locally. □

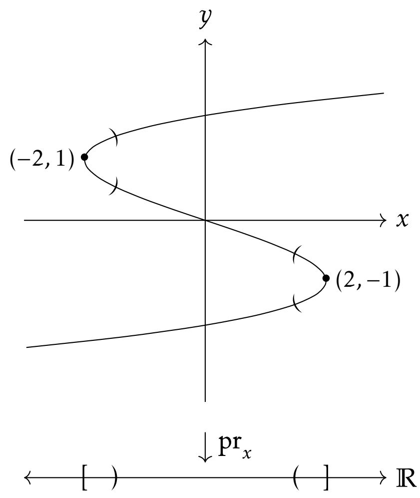


Figure 3.1 · The projection $\mathrm{pr}_x : V \to \mathbb{R}$ is not a local homeomorphism around the points $(\pm 2, \mp 1)$ since any open neighborhood is projected onto a non-open subset of $\mathbb{R}$.

In particular, by (i) of the above lemma, we can show that the local homeomorphisms over X form a category $\mathbf{LocalHomeo}_{/X}$, either as a full subcategory of $\mathbf{Top}_{/X}$ by (ii), or as the slice over X of the non-full subcategory **LocalHomeo** of **Top** where all maps are local homeomorphisms.

Example 3.1.4 · Let $X, Y \in \mathbf{Top}$ and $U \subseteq X$ open. We have the following list of examples:

(i) The inclusion $U \hookrightarrow X$ is a local homeomorphism.

(ii) Certain covering spaces $p : Y \to X$ are local homeomorphisms. work out

(iii) Consider the zero locus $V \subseteq \mathbb{R}^2$ of the polynomial $x - y^3 - 3y$. The projection on the x-axis $\mathrm{pr}_x : V \to \mathbb{R}$ is not a local homeomorphism as shown in Figure 3.1. However, after restricting to $V \setminus \{(\pm 2, \mp 1)\}$ it is.

(iv) Consider the line with two origins $\mathbb{R} \sqcup_{\mathbb{R}\setminus\{0\}} \mathbb{R}$ obtained by gluing two copies of the real line $\mathbb{R}$ along $\mathbb{R} \setminus \{0\}$; see Figure 3.2. There is a map q from the disjoint union $\mathbb{R} \sqcup \mathbb{R}$ to the line with two origins which sends the origin in the first copy of $\mathbb{R}$ to the one of the two origins in $\mathbb{R} \sqcup_{\mathbb{R}\setminus\{0\}} \mathbb{R}$ and the origin in the other copy to the other of the two origins. There is a further map p down to $\mathbb{R}$ which collapses the two origins. Both maps p and q are local homeomorphisms.

The line with two origins is not Hausdorff. In the second assignment, we will define a sheaf on $\mathbb{R}$ whose étalé space is the line with two origins, illustrating that the étalé space is usually not Hausdorff.

Lemma 3.1.5 · *Let F be a presheaf on X. Then the maps*

$$\coprod_{U \in \mathrm{Open}(X)} F(U) \times U \xrightarrow{q} \mathrm{sp}(F) \xrightarrow{p} X$$

are local homeomorphisms, where q is the quotient map and $p : [s, x]_U \mapsto x$.

Proof. *Clearly,* $p \circ q$ is a local homeomorphism. Thus, it suffices to show q is open on opens of the form $(\{s|_V\} \times V)_V$, where $s \in F(U)$, $V \subseteq U$ opens. For if $q((\{s|_V\} \times V)_V)$ is open, since $(s, x)_U \sim (s|_V, x)_V$ for all $x \in V$, $q((\{s\} \times V)_U) = q((\{s|_V\} \times V)_V)$ is open, so then q is open on all basis opens. Yet since q is a quotient map, this is equivalent to $O := q^{-1} \circ q((\{s\} \times U)_U)$ being

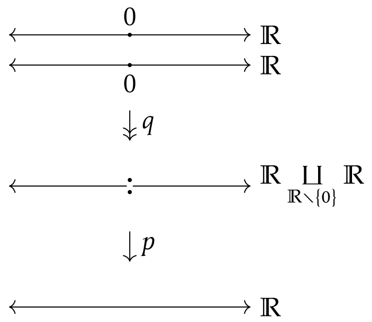


Figure 3.2 · Two local homeomorphisms related to the line with two origins

open. So let $(t,x)_V \in O$, then $x \in U$ and $(t,x)_V \sim (s,x)_U$. Hence there exists a $W \subseteq U \cap V$ such that $t|_W = s|_W$. Thus $(\{t\} \times W)_V \subseteq F(V) \times V$ is an open neighbourhood of $(t,x)_V$ in O. So q is open, therefore, by the previous lemma, p and q are local homeomorphisms. □

Of note, (iii) of Lemma 3.1.3 would not hold if we were to require g to be a quotient map instead of open. As a counterexample, let $I := [-1,1] \subseteq \mathbb{R}$. Let $Z = I \sqcup I$ be the disjoint union (with coordinates $(x, \pm 1)$), $Y = Z/\{(\pm 1, -1) \sim (\pm 1, 1)\} \cong S^1$ and $X = I$. Then while g is a quotient map and h a local homeomorphism, f is not a local homeomorphism (in the points $[(\pm 1, 1)]$).

3.2 Sheaf/space adjunction

'Let's go!'

We constructed functors

$$\mathbf{PSh}(X) \underset{h_{-/X}}{\overset{\mathrm{sp}}{\rightleftarrows}} \mathbf{Top}_{/X}$$

and showed that sp lands in $\mathbf{LocalHomeo}_{/X}$ and $h_{-/X}$ lands in $\mathbf{Sh}(X)$.

Theorem 3.2.1 · *There is an adjunction*

$$\mathbf{PSh}(X) \underset{h_{-/X}}{\overset{\mathrm{sp}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Top}_{/X}$$

which restricts to an adjoint equivalence

$$\mathbf{Sh}(X) \underset{h_{-/X}}{\overset{\mathrm{sp}}{\underset{\simeq}{\rightleftarrows}}} \mathbf{LocalHomeo}_{/X}$$

Example 3.2.2 · Let's check the adjunction for the sheaf $h_{U/X} = h_U = \mathrm{Hom}_{\mathrm{Open}(X)}(-, U)$. We showed in Example 2.3.5 that $\mathrm{sp}(h_{U/X}) = U$ in $\mathbf{Top}_{/X}$. On the other hand, the Yoneda Lemma 2.2.3 gives

$$\mathrm{Hom}_{\mathbf{PSh}(X)}(h_U, h_{Y/X}) \cong h_{Y/X}(U) = \mathbf{Top}_{/X}(U, Y).$$

Some formal nonsense if you want to feel fancy at dinner parties: sp is 'by definition' the left Kan extension of

$$\mathrm{Open}(X) \to \mathbf{Top}_{/X}, \quad U \mapsto (U \hookrightarrow X)$$

along the Yoneda embedding $h : \mathrm{Open}(X) \to \mathbf{PSh}(X)$.

Proof (of the adjunction in Theorem 3.2.1). Since $\mathrm{sp}(F)$ is the coequaliser of the diagram

$$\textstyle\coprod_{U\subseteq V} F(V)\times U \overset{a}{\underset{b}{\rightrightarrows}} \coprod_U F(U)\times U$$

we have

$$\mathbf{Top}_{/X}(\mathrm{sp}(F),Y)\cong\{f:\coprod_U F(U)\times U\to Y\mid fa=fb\}$$

and likewise

$$\begin{aligned}\mathbf{Top}_{/X}(\coprod_{U\subseteq X}F(U)\times U,Y)&\cong\prod_{U\subseteq X}\mathbf{Top}_{/X}(F(U)\times U,Y)\\&\cong\prod_{U\subseteq X}\mathrm{Map}(F(U),\mathbf{Top}_{/X}(U,Y))\\&=\prod_{U\subseteq X}\mathrm{Map}(F(U),h_{Y/X}(U))\end{aligned}$$

and

$$\mathbf{Top}_{/X}(\coprod_{U\subseteq V}F(V)\times U,Y)\cong\prod_{U\subseteq V}\mathrm{Map}(F(V),h_{Y/X}(U)).$$

The maps $-\circ a,-\circ b:\prod_U\mathrm{Map}(F(U),h_{Y/X}(U))\to\prod_{U\subseteq V}\mathrm{Map}(F(V),h_{Y/X}(U))$ are induced by

$$\begin{array}{ccccc}F(U)\times U&\xleftarrow{\mathrm{incl}^*\times\mathrm{id}}&F(V)\times U&\xrightarrow{\mathrm{id}\times\mathrm{incl}}&F(V)\times V\\ \Cup&&\Cup&&\Cup\\ (s|_U,x)&\longleftarrow\!\shortmid&(s,x)&\longmapsto&(s,x)\end{array}$$

so they are given by

$$\begin{array}{ccccc}\mathrm{Map}(F(U),h_{Y/X}(U))&\longrightarrow&\mathrm{Map}(F(V),h_{Y/X}(U))&&\mathrm{Map}(F(V),h_{Y/X})\\ \Cup&&\Cup&&\Cup\\ \alpha_U&\longmapsto&\alpha_U\circ r^F_{UV},\ r^{h_{Y/X}}_{UV}\circ\alpha_V&\longleftarrow\!\shortmid&\alpha_V\end{array}$$

for opens $U\subseteq V\subseteq X$. Hence $f=(\alpha_U)_U\in\prod_U\mathrm{Map}(F(U),h_{Y/X}(U))$ satisfies $fa=fb$ if and only if it is a natural transformation $F\Rightarrow h_{Y/X}$. $\square$

A fancy comment: we can express $\mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(F,G)$ as an end

$$\mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(F,G)\cong\int_{X\in\mathrm{ob}\,\mathscr{C}}\mathrm{Map}(F(X),G(X)).$$

Remark 3.2.3 · To check that adjoint functors

$$\mathscr{C}\underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\perp}}}}\mathscr{D}$$

are inverse equivalences of categories, you need to check that the *unit* and *counit* are isomorphisms. The unit is the natural transformation $\eta:\mathrm{id}_{\mathscr{C}}\Rightarrow UF$ whose component η_A is the transpose of the identity $\mathrm{id}_{FA}:FA\to FA$ under the adjunction $F\dashv U$. Dually, the counit is the natural transformation $\varepsilon:FU\Rightarrow\mathrm{id}_{\mathscr{D}}$ whose component ε_B is the transpose of the identity $\mathrm{id}_{UB}:UB\to UB$ under the adjunction.

Proof (of the restricted equivalence in Theorem 3.2.1). Let $\mathscr{F}$ be a sheaf on X. The unit $\eta_{\mathscr{F}}:\mathscr{F}\Rightarrow h_{\mathrm{sp}(\mathscr{F})/X}$ takes a section $s\in\mathscr{F}(U)$ to the section to $\mathrm{sp}(\mathscr{F})\to X$ given by

$$\begin{array}{ccccc}\coprod_V\mathscr{F}(V)\times V&\overset{q}{\twoheadrightarrow}&\mathrm{sp}(\mathscr{F})&\xrightarrow{p}&X\\ &\nwarrow&&\nwarrow{\scriptstyle(\eta_{\mathscr{F}})_U(s)}&\uparrow\\ &&&&U\end{array}$$

where the bottom map takes $x \in U$ to $(s, x)_U$. We need to show that the map

$$(\eta_{\mathcal{F}})_U : \mathcal{F}(U) \to \mathbf{Top}_{/X}(U, \mathrm{sp}(\mathcal{F}))$$

is a bijection.

For injectivity, suppose $(\eta_{\mathcal{F}})_U(s) = (\eta_{\mathcal{F}})_U(t)$ for $s, t \in F(U)$. Then $(s, x)_U \approx (t, x)_U$ for all $x \in U$, so there exists a neighbourhood $W_x \subseteq U$ of x on which s and t agree, that is, $s|_{W_x} = t|_{W_x}$. The family $(W_x)_{x \in U}$ covers U, so gluing gives $s = t$.

For surjectivity, suppose $f : U \to \mathrm{sp}(\mathcal{F})$ is a section. For any $x \in U$, pick a lift $(s_x, x)_{V_x} \in \coprod_V F(V) \times V$ of $f(x) \in \mathrm{sp}(\mathcal{F})$ along q, and set $W_x := (\{s_x\} \times V_x)_{V_x}$. Without loss of generality, we may assume V_x is contained in U (otherwise, intersect V_x with U). We showed in Lemma 3.1.5 that the restricted maps

$$W_x \xrightarrow[\cong]{q} q(X) \xrightarrow[\cong]{p} V_x$$

are homeomorphisms. The opens $U_x := f^{-1}(q(W_x))$ cover U as x runs through the points of U and f lifts to a map

$$f|_{U_x} : U_x \to F(V_x) \times V_x, \quad (s_x, x)_{V_x}$$

of the form $(\eta_{\mathcal{F}})_{U_x}(s_x)$. By injectivity, we get $s_x|_{U_x \cap U_y} = s_y|_{U_x \cap U_y}$ for all x and y, so they glue to a section $s \in \mathcal{F}(U)$ such that $f = (\eta_{\mathcal{F}})_U(s)$.

We conclude that the unit is an isomorphism for sheaves. We finish the proof next week by showing that the counit is an isomorphism for local homeomorphisms. □

LECTURE 4

Sheafification, monodromy

4.1 Sheaf/space adjunction (continued)

'Computers don't have noses.'

We still have to prove that the counit of the sheaf/space adjunction is an isomorphism for local homeomorphisms to prove the equivalence $\mathbf{Sh}(X) \simeq \mathbf{LocalHomeo}_{/X}$.

Proof (of the restricted equivalence in Theorem 3.2.1, continued). If $f : Y \to X$ is a local homeomorphism, we have to show that the counit

$$\varepsilon_{Y/X} : \operatorname{sp}(h_{Y/X}) \to Y$$

is an isomorphism in $\mathbf{LocalHomeo}_{/X}$ (or $\mathbf{Top}_{/X}$). Concretely, this map is given by

$$\begin{array}{ccc} \coprod_{U \subseteq X} \mathbf{Top}_{/X}(U, Y) \times U & \longrightarrow & Y \\ & \searrow\!\!\!\!\searrow & \nearrow_{\varepsilon_{Y/X}} \\ & \operatorname{sp}(h_{Y/X}) & \end{array}$$

where the top map sends $(s, x)_U$ to $s(x) \in Y$. This map indeed descends to the étalé space $\operatorname{sp}(h_{Y/X})$: if $x \in U \subseteq V$ and $s : V \to Y$ is a section, then $(s, x)_V \sim (s|_U, x)_U$, and both are sent to $s(x)$.

By Lemmas 3.1.3 and 3.1.5, the map $\operatorname{sp}(h_{Y/X}) \to Y$ is a local homeomorphism, so in particular an open map, so it suffices to check that it is bijective.

For injectivity, assume $\varepsilon_{Y/X}([s, x]_U) = \varepsilon_Y([t, y]_V)$. Then $x = y$ since everything is over X, and thus $s(x) = t(x)$ by assumption. Let S be an open neighbourhood of $s(x) = t(x)$ in Y such that f restricts to a homeomorphism $f|_S : S \to f(S)$, and set $W := s^{-1}(S) \cap t^{-1}(S)$. Observe that W contains x. Then $s|_W$ and $t|_W$ are both sections to the injective map $f|_S : S \hookrightarrow X$, so they agree. Hence $(s, x)_U \approx (s|_W, x)_W = (t|_W, x)_W \approx (t, x)_V$ and we conclude that $\varepsilon_{Y/X}$ is injective.

For surjectivity, let $y \in V$ and set $x = f(y) \in X$. Choose an open neighbourhood $V \subseteq Y$ of y such that f restricts to a homeomorphism $f|_V : V \to f(V) =: U$ onto an open subset. Write $s : U \to V$ for its inverse $f|_V^{-1}$; then $s : U \to V \subseteq Y$ is a section with $s(x) = y$, so $\varepsilon_{Y/X}([s, x]_U) = y$, showing that $\varepsilon_{Y/X}$ is surjective.

We conclude that the counit of the sheaf/space adjunction is an isomorphism for local homeomorphisms, finishing the proof of the equivalence of categories $\mathbf{Sh}(X) \simeq \mathbf{LocalHomeo}_{/X}$ of Theorem 3.2.1. □

Definition 4.1.1 · For a space X, the composite functor

$$(-)^\sharp := h_{-/X} \circ \operatorname{sp} : \mathbf{PSh}(X) \to \mathbf{Sh}(X)$$

is called *sheafification*. It comes with a natural map $F \Rightarrow F^\sharp$ for presheaves F on X given by the unit of the adjunction $\operatorname{sp} \dashv h_{-/X}$.

The following result can be seen as the universal property of sheafification.

Corollary 4.1.2 · *Sheafification is left adjoint to the inclusion* $\mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$. *That is, if F is a presheaf on X and $\mathcal{G}$ is a sheaf on X, then every map $F \Rightarrow \mathcal{G}$ of presheaves factors uniquely through the natural map $F \Rightarrow F^\sharp$.*

Proof. Compose the adjunction $\mathrm{sp} \dashv h_{-/X}$ and the equivalence $\mathbf{LocalHomeo}_{/X} \simeq \mathbf{Sh}(X)$. □

The unit of the adjunction is the map $F \Rightarrow F^\sharp$; it is an isomorphism if and only if F is a sheaf.

Example 4.1.3 · For the one-point space $*$, we have $\mathrm{Open}(*) = \{\varnothing \hookrightarrow *\}$, so

$$\mathbf{PSh}(X) = \mathbf{Fun}(\to, \mathbf{Set})$$

is the arrow category of **Set**. As we have seen in the homework, a presheaf F on the one-point space – that is, a map $F(*) \to F(\varnothing)$ – is a sheaf if and only if $F(\varnothing) = *$. Hence we have an equivalence $\mathbf{Sh}(X) \simeq \mathbf{Set}$. Of course, we also have $\mathbf{LocalHomeo}_{/*} \simeq \mathbf{Set}$ since having a local homeomorphism $X \to *$ implies X is discrete.

Example 4.1.4 · Let $S = \{0, 1\}$ be the Sierpiński space with

$$\mathrm{Open}(S) = \{\varnothing \hookrightarrow \{1\} \hookrightarrow S\}.$$

Using a similar argument as in the previous example, we see

$$\mathbf{PSh}(S) \simeq \mathbf{Fun}(\to\to, \mathbf{Set})$$

and

$$\mathbf{Sh}(S) \simeq \mathbf{Fun}(\to, \mathbf{Set}).$$

Exercise 4.1.5 · Check that sheafification on S is given by sending a presheaf F – that is, a pair of maps $F(S) \to F(\{1\}) \to F(\varnothing)$ – to the sheaf given by the map $F(S) \to F(\{1\})$.

4.2 Restriction of sheaves

'You can do this for different values of four, here we do it for five.'

Definition 4.2.1 · Let $U \subseteq X$ be open and let F be a presheaf on X. Then the *restriction* $F|_U$ of F to U is the restriction of the functor F along the inclusion $\mathrm{Open}(U) \hookrightarrow \mathrm{Open}(X)$.

Lemma 4.2.2 · *If $\mathcal{F}$ is a sheaf on X, then so is its restriction $\mathcal{F}|_U$ to an open $U \subseteq X$. Under the equivalence* $\mathbf{Sh}(X) \simeq \mathbf{LocalHomeo}_{/X}$, *restriction to U corresponds to sending a local homeomorphism $f : Y \to X$ to the pullback $f^{-1}(U) \to U$.*

In $\mathbf{LocalHomeo}_{/X}$, we have the subcategory $\mathbf{Cov}_{/X}$ of *covering spaces*: continuous maps $f : Y \to X$ such that X has an open cover $X = \bigcup_{i \in I} U_i$ and there are sets S_i such that $f^{-1}(U_i) \to U_i$ is isomorphic to $U_i \times S_I \to U_i$ (where S_i has the discrete topology) over U_i for all $i \in I$.

Pancake example

Remark 4.2.3 · We will not assume that the space Y in a covering space $Y \to X$ is (path) connected; this condition does appear in the literature with some authors.

Definition 4.2.4 · A sheaf $\mathcal{F}$ on a space X is *locally constant* if there is an open cover $X = \bigcup_{i \in I} U_i$ such that the restriction $\mathcal{F}|_{U_i}$ is a constant sheaf h_{S_i} for some set S_i for all $i \in I$.

We write $\mathbf{Sh}^{\mathrm{lc}}(X)$ for the full subcategory of $\mathbf{Sh}(X)$ on the locally constant sheaves.

Lemma 4.2.5 · *Under the equivalence* $\mathbf{Sh}(X) \simeq \mathbf{LocalHomeo}_{/X}$, *the locally constant sheaves correspond to covering spaces. In other words, this equivalence restricts to an equivalence*

$$\mathbf{Sh}^{\mathrm{lc}}(X) \simeq \mathbf{Cov}_{/X}.$$

The real content is the constant case: we have $\mathcal{F} \cong h_S = h_{X \times X/X}$ if and only if $\mathrm{sp}(\mathcal{F}) \cong X \times S$ over X.

Four-to-one cover of the circle example

4.3 Review of monodromy

'If you've ever been in a parking garage, you know what I mean.'

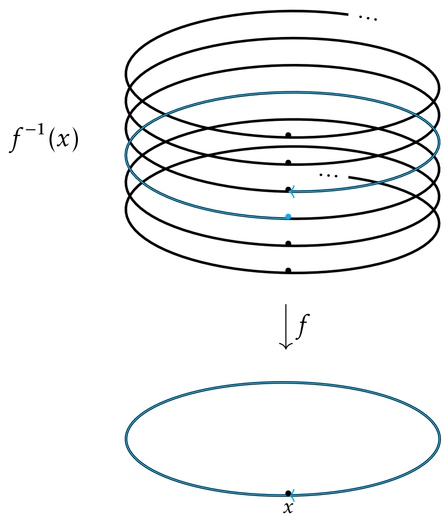


Figure 4.1 · Unique path lifting in the cover of the circle by the helix

Definition 4.3.1 · Let X be a topological space. Then X is

(i) *locally path connected* if every open neighbourhood $x \in U$ of any $x \in X$ contains a path connected open neighbourhood $x \in V \in U$.

(ii) *semi-locally path connected* if for every $x \in X$, there exists an open neighbourhood $x \in U$ such that the map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Definition 4.3.2 · Let X be a topological space and $x \in X$. The *fibre functor* is defined by

$$F_x : \mathbf{Cov}_{/X} \to \pi_1(X, x)\mathbf{Set}$$
$$(Y \xrightarrow{f} X) \mapsto f^{-1}(x),$$

where a loop $\gamma \in \pi_1(X, x)$ acts on $f^{-1}(x)$ by unique path lifting.

Theorem 4.3.3 (monodromy correspondence) · *Let X be a topological space and $x \in X$.*

(i) If X is path connected and locally path connected, then

$$F_x : \mathbf{Cov}_{/X} \to \pi_1(X, x)\mathbf{Set}$$

is fully faithful.

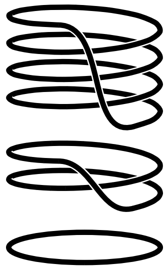


Figure 4.2 · Covering of the circle by a disjoint union of the four-to-one cover and the two-to-one cover.

(ii) If X is moreover semi-locally simply connected, then F_x is an equivalence of categories.

Proof (outline).

Maybe type up the details Marcel

- Any covering $f : Y \to X$ is locally path connected (easy). It is furthermore a disjoint union $\coprod_{i \in I} Y_i$ of path connected spaces Y_i [Munoo, Theorem 25.4].
- Any map $Y \xrightarrow{\varphi} Y' \in \mathbf{Cov}_{/X}$ is itself a covering [Munoo, Lemma 80.2(a)].
- F_x is faithful: if two maps

$$\begin{array}{ccc} Y & \overset{\varphi}{\underset{\varphi'}{\rightrightarrows}} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

  agree on $f^{-1}(x)$ then they agree: for $y \in Y$ arbitrary, choose a path γ starting at $f(y)$ and ending at x. By unique path lifting, this gives a path $\overline{\gamma}$ from y to y' for some $y' \in f^{-1}(x)$. Then $\varphi(\overline{\gamma})$ and $\varphi'(\overline{\gamma})$ are both lifts of γ to paths ending at $\varphi'(y') = \varphi(y')$ so by uniqueness they are the same path and their starting points are the same, i.e. $\varphi(y) = \varphi'(y)$.
- That F_x is full follows from the lifting lemma [Munoo, Lemma 79.1].
- F_x is essentially surjective if X is semi-locally simply connected: every $S \in \pi_1(X,x)\mathbf{Set}$ is $S \cong \cup_{i \in I} S_i$ where $\pi_1(X,x)$ acts transitively on S_i. Then

explain

$$S_i \cong \pi_1(X,x)/H_i$$

  for $H_i = \mathrm{Stab}(s_i)$ for any $s_i \in S_i$. There exists a covering $Y_i \xrightarrow{f_i} X$ with $\pi_1(Y_i, y_i) \hookrightarrow \pi_1(X,x)$, one subgroup H_i [Munoo, Theorem 82.1], so $S_i \cong F_x(Y_i \xrightarrow{f_i} X)$ and $S = F_x(\coprod_{i \in I} Y_i \to X)$.

□

Example 4.3.4 · Let $X = S^1$. Let Y_1 be the four-to-one covering of S^1, and let Y_2 be the double covering of S^1. Let $Y = Y_1 \coprod Y_2$. Then $Y \to X$ corresponds to the set $\{1,2,3,4,5,6\}$ where the single loop around the circle $1 \in \pi_1(S^1, x)$ acts by $(12)(3456)$.

Diagram below should have labels etc

The following diagram summarises the situation if X is a 'nice' space:

$$\begin{array}{ccccc} \mathbf{Sh}(X) & \xrightarrow{\simeq} & \mathbf{LocalHomeo}_{/X} & & \\ \uparrow & & \uparrow & & \\ \mathbf{Sh}^{\mathrm{lc}}(X) & \xrightarrow[\simeq]{} & \mathbf{Cov}_{/X} & \xrightarrow[\simeq]{} & \pi_1(X,x)\mathbf{Set} \end{array}$$

This diagram raises the question: what should there be in the top right spot? Towards the end of the course, we will give a partial answer which goes by the name of *exodromy*.

LECTURE 5

Pullback and pushforward

> 'There is a risk you might learn something – beware!'

The goal of this lecture is to define a pair of adjoint functors

$$\mathbf{Sh}(Y) \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Sh}(X)$$

for a continuous map $f : Y \to X$, called *pullback* and *pushforward*. The strategy will be to first define these operations for presheaves, and then restrict to sheaves. One of the functors will already send sheaves to sheaves, for the other we will postcompose the functor on the level of presheaves with sheafification.

5.1 Pushforward

A continuous map $f : Y \to X$ induces a functor $f^{-1} : \text{Open}(X) \to \text{Open}(Y)$ which sends an open set $U \subseteq X$ to the open set $f^{-1}(U) \subseteq Y$. If $U \subseteq V \subseteq X$ are open sets, then $f^{-1}(U) \subseteq f^{-1}(V) \subseteq Y$, so this is indeed functorial.

Definition 5.1.1 · The *pushforward* of a presheaf G on Y along f is the composite presheaf

$$f_*G : \text{Open}(X)^{\text{op}} \xrightarrow{f^{-1}} \text{Open}(Y)^{\text{op}} \xrightarrow{G} \mathbf{Set}$$

on X. Explicitly, the value of the pushforward f_*G on an open $U \subseteq X$ is $(f_*G)(U) = G(f^{-1}(U))$.

The pushforward functor $f_* : \mathbf{PSh}(Y) \to \mathbf{PSh}(X)$ is given by precomposition by the functor $f^{-1} : \text{Open}(X)^{\text{op}} \to \text{Open}(Y)^{\text{op}}$, and so is indeed seen to be functorial.

Lemma 5.1.2 · *If $\mathscr{G}$ is a sheaf on Y, then the pushforward $f_*\mathscr{G}$ of $\mathscr{G}$ along f is a sheaf on X. In particular, pushforward restricts to a functor $f_* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ on the level of sheaves.*

Proof. Let $U = \bigcup_{i\in I} U_i$ be an open cover in X. Then $f^{-1}(U) = \bigcup_{i\in I} f^{-1}(U_i)$ is an open cover in Y. Applied to this cover, the sheaf condition for $\mathscr{G}$ gives us an equaliser diagram

$$\mathscr{G}(f^{-1}(U)) \longrightarrow \prod_{i\in I} \mathscr{G}(f^{-1}(U_i)) \rightrightarrows \prod_{i,j\in I} \mathscr{G}(f^{-1}(U_i \cap U_j))$$

(where we have rewritten $f^{-1}(U_i) \cap f^{-1}(U_j) = f^{-1}(U_i \cap U_j)$), giving the sheaf condition for the pushforward $f_*\mathscr{G}$. □

5.2 Pullback

'Just sheafify the hell out of everything.'

The *pullback* of a sheaf on X along f should be a sheaf on Y. One way to approach the problem of constructing a presheaf on Y from a presheaf $\mathcal{F}$ on X, would be to try extending it along f^{-1} (and this is what we will attempt):

$$\begin{array}{ccc} \mathrm{Open}(X)^{\mathrm{op}} & \xrightarrow{\quad \mathcal{F} \quad} & \mathbf{Set} \\ & {\scriptstyle f^{-1}} \searrow \quad \nearrow {\scriptstyle ?} & \\ & \mathrm{Open}(Y)^{\mathrm{op}} & \end{array}$$

In general, there might not be such an on-the-nose extension, but we can approximate an extension by considering extensions *up to* a natural transformation. There could be many such approximations, so we want to consider the 'best possible approximation' in some suitable sense.

In category theory, the general problem of approximating an extension of a functor along another functor is studied using *Kan extensions*; we refer to [Rie16, Chapter 6] for an introduction of Kan extensions, whose definition we actually do not need to know for our current purposes. Here we will apply the general theory to our case to define the pullback of sheaves.[1]

Proposition 5.2.1 · *Left Kan extensions of presheaves on X (functors $\mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$) along the functor $f^{-1} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Open}(Y)^{\mathrm{op}}$ exist, and assemble into a functor*

$$\mathrm{Lan}_{f^{-1}} : \mathbf{PSh}(X) \to \mathbf{PSh}(Y)$$

which is left adjoint to the pushforward functor f_*:

$$\mathbf{PSh}(Y) \underset{f_*}{\overset{\mathrm{Lan}_{f^{-1}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{PSh}(X)$$

Moreover, the left Kan extension along f^{-1} of a presheaf F on X is given on opens $U \subseteq Y$ by

$$\mathrm{Lan}_{f^{-1}} F(U) = \operatorname*{colim}_{f^{-1}(W) \supseteq U} F(W) \tag{5.1}$$

where W ranges over opens of X (more precisely, the colimit diagram is indexed by the full subcategory of $\mathrm{Open}(X)$ on those W such that $f^{-1}(W) \supseteq U$, or equivalently $W \supseteq f(U)$).

Proof. This is a special case of [Rie16, Corollary 6.2.6]; the only nontrivial step in applying the general result to this special case is recognising that the comma category $f^{-1} \downarrow U$ for $U \in \mathrm{Open}(Y)^{\mathrm{op}}$ is the described index category, but this verification is elementary enough to be left to the reader. □

We would also like a description of what $\mathrm{Lan}_{f^{-1}}$ does on maps. By a computation of the colimit in (5.1), an element of $\mathrm{Lan}_{f^{-1}} F(U)$ is given by $[s]_W$ for some section $s \in F(W)$ for some $W \supseteq f(U)$, where $[s]_W = [s']_{W'}$ if and only if there exists an open $W'' \subseteq W \cap W'$ containing $f(U)$ such that $s|_{W''} = s'|_{W''}$. Unravelling the construction in [Rie16, Theorem 6.2.1], we see that the map induced by opens $U \subseteq V$ in Y is given by

$$\operatorname*{colim}_{f^{-1}(W) \supseteq V} F(W) \to \operatorname*{colim}_{f^{-1}(W) \supseteq U} F(W), \quad [s]_W \mapsto [s]_W.$$

We may now define the pullback as follows:

[1] One can also construct and prove everything by hand, as was done in class, but this is rather tedious. We wish to illustrate with the following exposition that all the arguments will be completely formal.

Definition 5.2.2 · The *pullback* $f^{\circledast}F$ of a presheaf F on X along f is the left Kan extension of F along f^{-1}. The pullback functor is denoted $f^{\circledast} := \mathrm{Lan}_{f^{-1}} : \mathbf{PSh}(X) \to \mathbf{PSh}(Y)$.

We have defined the pullback functor in such a way that it is left adjoint to the pushforward of presheaves.

We use the circled asterisk $\circledast$ in the notation for the pullback of presheaves to distinguish it from the pullback of sheaves, which we will define momentarily, and for which we use a normal asterisk.[2] Unlike the pushforward, namely, the pullback of presheaves does not directly restrict to sheaves; that is to say, there are sheaves which are sent by $f^{\circledast}$ to a presheaf which does not satisfy the sheaf condition, as witnessed by the following counterexample:

Example 5.2.3 · Let Y be the two-point space with the discrete topology, and let X be the one-point space, with the unique map $f : Y \to X$. We can easily put a sheaf on X, for example by defining $\mathcal{F}(\varnothing) = *$ and $\mathcal{F}(X) = \{42, 43, 44\}$. The pullback in presheaves at any point $\{*\} \subseteq Y$ is $f^{\circledast}(\{*\}) = \mathrm{colim}_{f^{-1}(W) \supseteq *} \mathcal{F}(W) = \mathcal{F}(X) = \{42, 43, 44\}$. But the pullback in presheaves at Y itself is also $\mathcal{F}(X) = \{42, 43, 44\}$. Check that the sheaf condition doesn't hold for the cover of Y given by two opens, one containing precisely each point: the gluing condition on $\{42, 43, 44\} \times \{42, 43, 44\}$ is void because the points do not intersect.

Definition 5.2.4 · The *pullback* $f^*\mathcal{F}$ of a sheaf $\mathcal{F}$ on X along a continuous map $f : Y \to X$ is the sheaf

$$f^*\mathcal{F} := (f^{\circledast}\mathcal{F})^{\sharp}$$

on Y, the sheafification of the presheaf pullback of $\mathcal{F}$. The pullback functor of sheaves is thus the composite

$$f^* : \mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X) \xrightarrow{f^{\circledast}} \mathbf{PSh}(Y) \xrightarrow{(-)^{\sharp}} \mathbf{Sh}(Y).$$

Note that the definition of the pullback also makes sense for presheaves; we also denote the functor $\mathbf{PSh}(X) \to \mathbf{Sh}(Y)$ sending a presheaf F to the pullback $f^*F = (f^{\circledast})^{\sharp}$ by f^*.

Proposition 5.2.5 · *Pushforward and pullback of sheaves along f define an adjunction*

$$\mathbf{Sh}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Sh}(X) \quad (f^* \dashv f_*)$$

Proof. Compose the adjunctions

$$\begin{array}{ccccc} & & \xleftarrow{\quad f^* \quad} & & \\ \mathbf{Sh}(Y) & \underset{\hookrightarrow}{\overset{(-)^{\sharp}}{\leftarrow}} & \mathbf{PSh}(Y) & \underset{f_*}{\overset{f^{\circledast}}{\rightleftarrows}} & \mathbf{PSh}(X) \\ & & & & \uparrow \\ & & \xrightarrow{\quad f_* \quad} & & \mathbf{Sh}(X) \end{array}$$

of Corollary 4.1.2 and Proposition 5.2.1. □

Between the sets $\mathrm{Hom}_{\mathbf{Sh}(Y)}(f^*\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, f_*\mathcal{G})$ which are naturally isomorphic by the adjunction, there is also an 'intermediate' set of maps, called f*-maps*; see [Sta24, Lemma 008K].

From the fact that the pushforward already sends sheaves to sheaves (so its definition 'doesn't need sheafification'), we obtain the following corollary, showing that pulling back the sheafification of a presheaf is the same as pulling back the presheaf.

[2]In class, we used the notation $f^{\mathrm{pre},*}$ for $f^{\circledast}$, which we do not find very elegant.

Corollary 5.2.6 · *Let $f : Y \hookrightarrow X$ be a continuous map. Then the functors*

$$\mathbf{PSh}(X) \xrightarrow{f^*} \mathbf{Sh}(Y) \qquad \text{and} \qquad \mathbf{PSh}(X) \xrightarrow{(-)^\sharp} \mathbf{Sh}(X) \xrightarrow{f^*} \mathbf{Sh}(Y)$$

are naturally isomorphic.

Proof. By definition, the former functor is the composite

$$\mathbf{PSh}(X) \underset{f_*}{\overset{f^\circledast}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{PSh}(*) \underset{}{\overset{(-)^\sharp}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Sh}(*)$$

of the presheaf pullback along f and sheafification, which have right adjoints given by respectively the pushforward along f and the full subcategory inclusion by Proposition 5.2.1 and Corollary 4.1.2. The latter functor has a right adjoint

$$\mathbf{PSh}(X) \overset{(-)^\sharp}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}}} \mathbf{Sh}(X) \underset{f_*}{\overset{f^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Sh}(*)$$

given by pushforward along f followed by the full subcategory inclusion by Corollary 4.1.2 and Proposition 5.2.5. These two right adjoint are equal by definition, so we find the required natural isomorphism. □

The following corollary roughly says that pushforward and pullback are 'functorial in the map' (respectively co- and contravariantly).[3]

Corollary 5.2.7 · *For any space X, we have $(\mathrm{id}_X)_* = \mathrm{id}_{\mathbf{Sh}(X)} \cong (\mathrm{id}_X)^*$, and for any maps $f : Z \to Y$ and $g : Y \to X$ we have $(gf)_* = g_* f_* : \mathbf{Sh}(Z) \to \mathbf{Sh}(X)$ and $(gf)^* \cong f^* g^* : \mathbf{Sh}(X) \to \mathbf{Sh}(Z)$.*

Note that the identities for the pushforward hold on-the-nose, whereas we only prove the identities for the pullback up to natural isomorphism.

Proof. The identities for the pushforward follow directly from the definition since $(gf)^{-1}(U) = f^{-1}(g^{-1}(U))$. The identities for the pullback follow from those for the pushforward and the adjunction of Proposition 5.2.5: we have adjunctions $(gf)^* \dashv (gf)_*$ and $f^* g^* \dashv g_* f_*$ and the right adjoints in these adjunctions are equal. □

5.3 Stalks, germs, skyscrapers

Definition 5.3.1 · Let $i_x : \{x\} \hookrightarrow X$ be the inclusion of a point in a space. Then $i_x^* \mathcal{F}$ is the *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at x; this is a sheaf on a point, so equivalently a set by Example 4.1.3 (the value of the sheaf on the whole space). Unravelling definitions, we have

$$\mathcal{F}_x = \operatorname*{colim}_{U \ni x} \mathcal{F}(U).$$

An element of the stalk at x is represented by a *germ* $[s]_U$ for $U \ni x$ and $s \in \mathcal{F}(U)$, where $[s]_U = [t]_V$ if and only if there exists an open neighbourhood $W \subseteq U \cap V$ of x such that $s|_W = t|_W$.

Note that the stalk $\mathcal{F}_x$ is the fibre of $\mathrm{sp}(X) \to X$ over x.

Remark 5.3.2 · Corollary 5.2.6 tells us that we can compute the stalks of the sheafification of a presheaf by computing the stalks of the presheaf itself.

[3] This statement can probably be made precise in terms of 2-categories.

Example 5.3.3 (orientation sheaf of a manifold) · Let M be an n-dimensional topological manifold, by which we mean a Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$ (i.e., every point of M admits a neighbourhood homeomorphic to $\mathbb{R}^n$). For a subset $K \subseteq M$, we write

$$H_k(M \mid K; R) := H_k(M, M \smallsetminus K; R)$$

for the kth homology of M relative to $M \smallsetminus K$ with coefficients in a commutative ring R.

A *local R-orientation* at a point $x \in M$ is a generator μ_x of $H_n(M \mid x; R) \cong R$ (note that we take homology in degree n, the dimension of the manifold M). That $H_n(M \mid x; R)$ is isomorphic to R (as R-modules) follows from excision; choosing a generator corresponds exactly to choosing such an isomorphism. An *R-orientation* of M is a family of local orientations $(\mu_x)_{x \in M}$ with the property that every point $y \in M$ admits a compact neighbourhood K and an element $\mu_K \in H_n(M \mid K; R)$ such that μ_K restricts to μ_x for any $x \in K$ under the map $H_n(M \mid K; R) \to H_n(M \mid x; R)$ induced by the inclusion $M \smallsetminus K \to M \smallsetminus \{x\}$.

There is a sheaf $\mathcal{O}_M$ of R-modules, the *orientation sheaf* of M, defined by

$$\mathcal{O}_M(U) := H_n(M \mid U; R)$$

for $U \subseteq M$ open. The restriction map for opens $U \subseteq V \subseteq M$ is induced by the inclusion of pairs $(M, M \smallsetminus V) \hookrightarrow (M, M \smallsetminus U)$. The stalk of the orientation sheaf $\mathcal{O}_M$ at $x \in M$ is the R-module $H_n(M \mid x; R)$.

Remark 5.3.4 · One can show (as a generalisation of an exercise in Homework 3) that the étalé space of the pullback can be obtained as a fibre product. More precisely, given $f : Y \to X$, the diagram

$$\begin{array}{ccc} \mathrm{sp}(f^*\mathcal{F}) & \longrightarrow & \mathrm{sp}(\mathcal{F}) \\ \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is a pullback square. This is used in [MM94, § II.9] to define f^*.

Given a map $\alpha : \mathcal{F} \to \mathcal{G}$ between sheaves on a space X and a point $x \in X$, there is an induced map on stalks $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$, because $\mathcal{G}_x$ is a cocone over

$$\{\mathcal{F}(U) \mid U \in \mathrm{Open}(X), x \in U\},$$

of which $\mathcal{F}_x$ is a colimit.

Lemma 5.3.5 · *Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a map in $\mathbf{Sh}(X)$. Then α is an isomorphism of sheaves if and only if $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is a bijection for all $x \in X$.*

Proof. The map $\mathrm{sp}(\alpha) : \mathrm{sp}(\mathcal{F}) \to \mathrm{sp}(\mathcal{G})$ is a local homeomorphism over X, because the maps $\mathrm{sp}(\mathcal{F}) \to X$ and $\mathrm{sp}(\mathcal{G}) \to X$ are local homeomorphisms. In particular, $\mathrm{sp}(\alpha)$ is an open map. So it is invertible if and only if it is a bijection. Check that this is equivalent to requiring that the map on the fibres over x is a bijection for all $x \in X$. Since the stalks $\mathcal{F}_x$ and $\mathcal{G}_x$ are exactly these fibres, we are done. □

The next lemma shows that the pullback of a constant sheaf along any map is again constant.

Lemma 5.3.6 · *Let $f : Y \to X$ be a continuous map and let S be a set. Then $f^*\underline{S}_X \cong \underline{S}_Y$.*

Proof. It suffices to show the lemma holds for the special case of the maps $p_W : W \to *$ from a space to a point. The general case will then follow. Indeed, then

$$f^*\underline{S}_X \cong f^* p_X^* \underline{S}_* \cong (f \circ p_X)^* \underline{S}_* = p_Y^* \underline{S}_* \cong \underline{S}_Y$$

by Corollary 5.2.7. For the special case, the presheaf pullback on a nonempty open becomes constant:

$$p_W^{\circledast} \underline{S}_*(U) = \operatorname*{colim}_{V \supseteq p_W(U)} \underline{S}_*(V) \cong \underline{S}_*(*) = S.$$

On the empty set, we have $p_W^{\circledast}(\varnothing) = *$. Its sheafification $p_W^* \underline{S}_*$ is then $\underline{S}_W$. (Verify that sheafification is unaffected by the value of a presheaf on the empty set.) □

Definition 5.3.7 · Let $\{x\} \to X$ be the inclusion of a point $x \in X$ and let S be a set. The *skyscraper sheaf* at x with value S is $i_{x,*}\underline{S}$, the pushforward of the constant sheaf. We denote it $i_{*,S}$.

Example 5.3.8 · Let $\{0\} \to \mathbb{R}$ be the inclusion of zero into the real numbers. Let $S = \{a, b\}$. Then the skyscraper sheaf is

$$i_{*,S}(U) = S(U \cap 0) = \begin{cases} \{a, b\} & \text{if } 0 \in U \\ * & \text{if } x \notin U. \end{cases}$$

In Homework 2, we saw that the étalé space of this sheaf is the line with two origins.

LECTURE 6

Sheaves of abelian groups

6.1 Sheaves of abelian groups

> 'Why is he erasing an empty board,
> you might ask.'

Almost everything we have done so far works for sheaves with algebraic structure, for example:

- Sheaves of abelian groups ($\rightsquigarrow$ homological algebra)
- Sheaves of commutative rings (algebraic geometry)
- Sheaves of commutative monoids (useful in logarithmic geometry (?))

You can also do more general things:

- Sheaves of topological rings
- Sheaves of Banach algebras
- Sheaves of categories (*stacks in categories*)
- Sheaves of groupoids (*stacks*)
- Sheaves of homotopy types/spaces/anima/∞-groupoids (∞-*stacks*)

We should warn that the sheaf condition should be modified for higher-categorical sheaves.

Definition 6.1.1 · Let $\mathscr{C}$ be a category. A *presheaf on* $X \in \mathbf{Top}$ *with values in* $\mathscr{C}$ is a functor $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathscr{C}$. If $\mathscr{C}$ is complete (or at least the products of the sheaf condition), then a sheaf on X with values in $\mathscr{C}$ is a presheaf such that the sheaf condition holds in $\mathscr{C}$.

We denote the categories of such objects $\mathbf{PSh}(X, \mathscr{C})$ and $\mathbf{Sh}(X, \mathscr{C})$ respectively. In the case $\mathscr{C}$ is $\mathbf{Ab}$, we may use the notation $\mathbf{PAb}(X) := \mathbf{PSh}(X, \mathbf{Ab})$ and $\mathbf{Ab}(X) := \mathbf{Sh}(X, \mathbf{Ab})$.

We will need a description of limits in $\mathbf{Ab}$.

Lemma 6.1.2 ([Mac71, Section v.1]) · *The forgetful functor* $\mathbf{Ab} \to \mathbf{Set}$ *creates limits.*

Example 6.1.3 · The product of a family $(A_i)_{i\in I}$ of abelian groups is given by the set $\prod_{i\in I} A_i$ with coordinatewise addition. Equalisers in $\mathbf{Ab}$ (and in fact in any additive category with all kernels) are given by

$$\mathrm{eq}(f, g) = \{ a \in A \mid f(a) = g(a) \} = \ker(f - g).$$

However, colimits are not the same as in $\mathbf{Set}$. The coproduct of a family $(A_i)_{i\in I}$ of abelian groups is the direct sum

$$\bigoplus_{i\in I} A_i = \{ (a_i)_{i\in I} \in \prod_{i\in I} A_i \mid \#\{ i \in I \mid a_i \neq 0 \} < \infty \},$$

and coequalisers in **Ab** are given by $\mathrm{coeq}(f,g) = \mathrm{coker}(f-g)$.

Corollary 6.1.4 · *A presheaf* $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Ab}$ *is a sheaf if and only if the composite*

$$\mathrm{Open}(X)^{\mathrm{op}} \xrightarrow{F} \mathbf{Ab} \xrightarrow{U} \mathbf{Set}$$

is a sheaf, where $U : \mathbf{Ab} \to \mathbf{Set}$ *is the forgetful functor.*

Definition 6.1.5 · A small category $\mathcal{I}$ is *filtered* if $\mathcal{I} \neq \varnothing$ and the following conditions hold:

(i) For all $i, j \in \mathcal{I}$ there exists a $k \in \mathcal{I}$ and arrows $i \to k \leftarrow j$

(ii) For $u, v : i \to j$ in $\mathcal{I}$ there exists a $k \in \mathcal{I}$ and $w : j \to k$ such that $wu = wv$.

Dually, there is a notion of *cofiltered category*.

Example 6.1.6 · A poset P always satisfies (ii), so it is filtered if and only if it is nonempty and every pair of elements has an upper bound. If I is a set, then the poset $\mathcal{P}_{\mathrm{fin}}(I) := \{ J \subseteq I \mid I \text{ finite} \}$ of finite subsets of I ordered by inclusion is filtered, since $\mathcal{P}_{\mathrm{fin}}(I)$ contains the empty set and two finite subsets $J, J' \subseteq I$ have an upper bound $J \cup J'$.

Exercise 6.1.7 · Show that a small category $\mathcal{I}$ is filtered if and only if every finite diagram $D : \mathcal{J} \to \mathcal{I}$ in $\mathcal{I}$ has a cocone.

Definition 6.1.8 (Only the notation was introduced in the lecture) · A *filtered colimit*, denoted $\varinjlim$ colim, is the colimit of a functor $F : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a filtered category. A *cofiltered limit*, denoted $\varprojlim$ lim, is the limit of a functor $F : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a cofiltered category.

Lemma 6.1.9 ([Mac71, Section IX.1]) · *The forgetful functor* $U : \mathbf{Ab} \to \mathbf{Set}$ *creates filtered colimits.*

Example 6.1.10 · The following are examples of filtered limits and colimits, in some cases the indexing category is omitted in the notation.

- We saw that
$$\bigoplus_{i\in I} A_i \cong \underset{J\in\mathcal{P}_{\mathrm{fin}}(I)}{\underrightarrow{\mathrm{colim}}} \oplus_{j\in J} A_j.$$
- The filtered colimit of the rings $\frac{1}{n}\mathbb{Z}$ where the diagram is indexed by the divisibility poset is
$$\underset{n}{\underrightarrow{\mathrm{colim}}} \frac{1}{n}\mathbb{Z} = \mathbb{Q}.$$
- The filtered colimit of the field extensions of $\mathbb{Q}$ is
$$\underset{\mathbb{Q}\to K \text{ finite}}{\underrightarrow{\mathrm{colim}}} K = \overline{\mathbb{Q}}.$$
- The filtered limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$ for p a prime is the ring
$$\underleftarrow{\lim}_n \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$$
of the p-adic integers.
- The stalk of a sheaf $\mathcal{F}$ at a point x is a filtered colimit:
$$\mathcal{F}_x = \underset{U\ni x}{\underrightarrow{\mathrm{colim}}} \mathcal{F}(U).$$
Indeed the indexing category $\{ U \in \mathrm{Open}(X)^{\mathrm{op}} \mid x \in U \}$ is filtered: if $x \in U, V \subseteq X$ are open then $x \in U \cap V$.
- Likewise, the pullback of a presheaf $\mathcal{F}$ is given by
$$f^{\circledast}\mathcal{F}(V) = \underset{f(V)\subseteq U}{\underrightarrow{\mathrm{colim}}} \mathcal{F}(U),$$
a filtered colimit.

6.2 Abelian group objects

> 'I'm only familiar with this for ∞-categories, not for 1-categories.'

Definition 6.2.1 · Let $\mathscr{C}$ be a category with finite products. An *abelian group object* in $\mathscr{C}$ is a quadruple $(A, m, i, 0)$ of an object $A \in \mathscr{C}$ and morphisms

$$m : A \times A \to A, \quad i : A \to A, \quad 0 : * \to A$$

such that the following diagrams commute:

(i) Associativity:

$$\begin{array}{ccc} A\times A\times A & \xrightarrow{m\times \mathrm{id}} & A\times A \\ {\scriptstyle \mathrm{id}\times m}\downarrow & & \downarrow{\scriptstyle m} \\ A\times A & \xrightarrow[m]{\qquad} & A \end{array}$$

(ii) Commutativity:

$$\begin{array}{ccc} A\times A & \xrightarrow{\quad\mathrm{swap}\quad} & A\times A \\ & {\scriptstyle m}\searrow \qquad \swarrow{\scriptstyle m} & \\ & A & \end{array}$$

(iii) Identity:

$$\begin{array}{ccccc} A\times * & \xrightarrow{\mathrm{id}\times 0} & A\times A & \xleftarrow{0\times \mathrm{id}} & *\times A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ A & \xrightarrow[\mathrm{id}]{\qquad} & A & \xleftarrow[\mathrm{id}]{\qquad} & A \end{array}$$

(iv) Inverse:

$$\begin{array}{ccccccccc} * & & \longleftarrow & & A & & \longrightarrow & & * \\ {\scriptstyle 0}\downarrow & & & & \downarrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle 0} \\ A & \xleftarrow[m]{} & A\times A & \xleftarrow[i\times \mathrm{id}]{} & A\times A & \xrightarrow[\mathrm{id}\times i]{} & A\times A & \xrightarrow[m]{} & A \end{array}$$

Write $\mathbf{Ab}(\mathscr{C})$ for the *category of abelian group objects* in $\mathscr{C}$. A map $f : (A, m, i, 0) \to (B, m', i', 0')$ in $\mathbf{Ab}(\mathscr{C})$ is a map $f : A \to B$ in $\mathscr{C}$ such that $f \circ m = m' \circ (f \times f)$, that is, such that f commutes with the group operation m.

Using the diagrams above, one can show that a map of abelian group objects in $\mathscr{C}$ also commutes with inverses and the unit.

Remark 6.2.2 · The category **Ab** of abelian groups is precisely the category $\mathbf{Ab}(\mathbf{Set})$ of abelian group objects in **Set**.

Lemma 6.2.3 · *We have the following equivalences of categories:*

(i) $\mathbf{Ab}(\mathbf{PSh}(X)) \cong \mathbf{PAb}(X)$

(ii) $\mathbf{Ab}(\mathbf{Sh}(X)) \cong \mathbf{Ab}(X)$

Proof (sketch). (i) A presheaf $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Ab}$ consists of $F(U) \in \mathbf{Ab}$ for all $U \in \mathrm{Open}(X)^{\mathrm{op}}$ with restriction maps $r_{UV} : F(V) \to F(U)$ in $\mathbf{Ab}$. This means that the group operation $F(U) \times F(U) \to F(U)$ is natural, that is, if $U \subseteq V$ then

$$\begin{array}{ccc} F(V) \times F(V) & \xrightarrow{m_V} & F(V) \\ {\scriptstyle r_{UV} \times r_{UV}} \downarrow & & \downarrow {\scriptstyle r_{UV}} \\ F(U) \times F(U) & \xrightarrow[m_U]{} & F(U) \end{array}$$

So $m : F \times F \to F$ is natural. Likewise, 0 and i are natural transformations (follow your nose). Conversely, an abelian object in $\mathbf{PSh}(X)$ is a presheaf of abelian groups (why?).

(ii) By Corollary 6.1.4, a presheaf $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Ab}$ is a sheaf if and only if the composite $U \circ F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf, where $U : \mathbf{Ab} \to \mathbf{Set}$ is the forgetful functor. Thus for any sheaf $\mathcal{F} \in \mathbf{Ab}(X)$, $U \circ \mathcal{F}$ is a sheaf of sets which has abelian group object structure by (i). Conversely, $(\mathcal{F}, m, i, 0) \in \mathbf{Ab}(\mathbf{Sh})$ is the data of a sheaf of abelian groups, since it is a presheaf of abelian groups by (i) and thus a sheaf since it is a sheaf on sets. □

Remark 6.2.4 · There is a 'tensor product' $\otimes$ on presentable categories for which we (probably) have $\mathbf{Ab}(X) \simeq \mathbf{Sh}(X) \otimes \mathbf{Ab}$.

Remark 6.2.5 · We can do everything we know for sheaves of abelian groups as well:

- We have an adjunction:

$$\mathbf{PAb}(X) \underset{h_{-/X}}{\overset{\mathrm{sp}}{\rightleftarrows}} \mathbf{Ab}(\mathbf{Top}_{/X})$$

  which restricts to an equivalence:

$$\mathbf{Ab}(X) \underset{h_{-/X}}{\overset{\mathrm{sp}}{\rightleftarrows}} \mathbf{Ab}(\mathbf{LocalHomeo}_{/X})$$

- Monodromy:

$$\mathbf{Ab}(\mathbf{Sh}^{\mathrm{lc}}(X)) \simeq \mathbf{Rep}_{\mathbb{Z}}(\pi_1(X, x)),$$

  where $\mathbf{Rep}_{\mathbb{Z}}(G)$ denotes the category of G-representations in $\mathbf{Mod}_{\mathbb{Z}}$. We may also denote $\mathbf{Ab}(\mathbf{Sh}^{\mathrm{lc}}(X))$ by $\mathbf{Ab}^{\mathrm{lc}}(X) \simeq \mathbf{Mod}_{\mathbb{Z}[\pi_1(X,x)]}$, where $\mathbb{Z}[\pi_1(X, x)]$ is the group algebra of $\pi_1(X, x)$.

- Pushforward and pullback: for $f : Y \to X$ we have adjunctions

$$\mathbf{PAb}(Y) \underset{f_*}{\overset{f^{\circledast}}{\leftrightarrows}} \mathbf{PAb}(X)$$

  and

$$\mathbf{Ab}(Y) \underset{f_*}{\overset{f^{*}}{\leftrightarrows}} \mathbf{Ab}(X)$$

- All the functors $f_*, f^{\circledast}, f^*, (-)^{\sharp}, h_{-/X}$ commute with the forgetful functors (see [Sta24, Lemma 0085] for a hands-on proof that $(-)^{\sharp}$ does).

- A map $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Ab}(X)$ is an isomorphism if and only if the induced map $f_x : \mathcal{F}_x \to \mathcal{G}_x$ on stalks is an isomorphism for all $x \in X$.

In order to do homological algebra for sheaves of abelian groups, we need a notion of *exactness* for sequences of sheaves of abelian groups. Since the image presheaf of map of sheaves is in general not a sheaf itself, we define exactness in $\mathbf{Ab}(X)$ via stalks.

Definition 6.2.6 · A sequence

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

of sheaves of abelian groups on a space X is *exact at* $\mathcal{G}$ if the induced sequence

$$\mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x \xrightarrow{g_x} \mathcal{H}_x$$

of stalks is exact at $\mathcal{G}_x$ for all points $x \in X$.

Remark 6.2.7 · Incidentally, the reason to define exactness via the stalks – that the image presheaf is not a sheaf – is also the reason the global sections functor (evaluation in the global sections) is generally only left exact, and thus gives rise to a right derived functor. Right deriving the global sections functor on a sheaf $\mathcal{F} \in \mathbf{Ab}(X)$ allows us to compute sheaf cohomology, denoted $H^i(X; \mathcal{F})$.

Example 6.2.8 · Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathbf{Ab}$. Then the sequence

$$0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0$$

of constant sheaves is exact. Indeed, $(\underline{Z})_x \cong Z$ for any $x \in X$ and any abelian group or set Z, since this is the fibre of $Z \times X \xrightarrow{\mathrm{pr}_X} X$ at $x \in X$.

Example 6.2.9 · Let $\iota : \{0\} \hookrightarrow \mathbb{R}$, $A \in \mathbf{Ab}$. Then $\iota_* A$ has etale space,

draw line with #A origins

LECTURE 7

Limits and colimits of (pre)sheaves

Definition 7.0.1 · A *diagram* in a category $\mathscr{C}$ is a functor $D : \mathcal{J} \to \mathscr{C}$ from a small category $\mathcal{J}$.

In this lecture, we will show that every diagram $D : \mathcal{J} \to \mathbf{PSh}(X)$ or $D : \mathcal{J} \to \mathbf{PAb}(X)$ has a limit and colimit, and we will describe them explicitly.

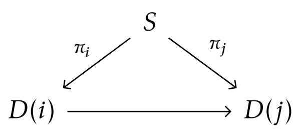
7.1 Limits and colimits of sets

We will begin by treating limits and colimits in **Set**. For a more complete treatment of this topic, we refer the reader to [Rie16, § 3.2].

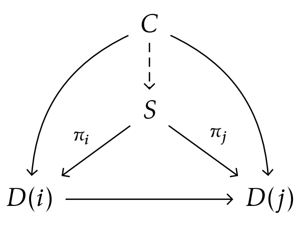
Proposition 7.1.1 · *The limit of* $D : \mathcal{J} \to$ **Set** *exists and is given by*

$$\lim_{i \in \mathcal{J}} D(i) = \{(a_i)_{i \in \mathcal{J}} \in \prod_{i \in \mathcal{J}} D(i) \mid D(\varphi)(a_i) = a_j \textit{ for all } \varphi : i \to j \textit{ in } \mathcal{J}\}$$

Proof. Denote our candidate limit set S. There are maps $S \xrightarrow{\pi_i} D(i)$ for all $i \in \mathcal{J}$. For any $\varphi : i \to j$ in $\mathcal{J}$ the diagram



commutes. This commutativity is directly by construction of our set S. Given any cone $(C \xrightarrow{\pi_i} D(i))_{i \in \mathcal{J}}$ there exists a unique map of cones to $(S \xrightarrow{\pi_i} D(i))_{i \in \mathcal{J}}$ making the diagram



commute. It is given by $c \mapsto (\varphi_i(c))_{i \in \mathcal{J}}$.

□

Remark 7.1.2 · The limit of a diagram $D : \mathcal{J} \to$ **Set** is the equalizer of

$$\prod_{i \in \mathcal{J}} D(i) \rightrightarrows \prod_{\varphi : i \to j} D(j)$$

with maps

$$(a_i)_{i\in\mathcal{J}} \mapsto (a_j)_{\varphi:i\to j}$$
$$(a_i)_{i\in\mathcal{J}} \mapsto (D(\varphi)(a_i))_{\varphi:i\to j}.$$

Corollary 7.1.3 · *For any locally small category $\mathscr{C}$ we have*

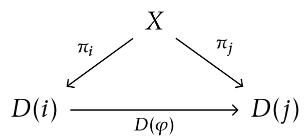
$$\mathrm{Hom}_{\mathscr{C}}(X, \lim_{i\in\mathcal{J}} D(i)) \cong \lim_{i\in\mathcal{J}} \mathrm{Hom}_{\mathscr{C}}(X, D(i)),$$

$$\mathrm{Hom}_{\mathscr{C}}(\operatorname{colim}_{i\in J} D(i), Y) \cong \lim_{i\in\mathcal{J}} \mathrm{Hom}_{\mathscr{C}}(D(i), Y).$$

Proof. (Sketch of the first isomorphism) To give a map $X \to \lim D(i)$ is to give a cone

$$(X \to D(i))_{i\in\mathcal{J}};$$

this means giving maps $\pi_i \in \mathrm{Hom}_{\mathscr{C}}(X, D(i))$ such that for all $\varphi : i \to j$ in $\mathcal{J}$ the diagram



commutes. That is, it is to give a collection $(\pi_i)_{i\in\mathcal{J}} \in \prod_{i\in\mathcal{J}} \mathrm{Hom}(X, D(i))$ such that $\mathrm{Hom}_{\mathscr{C}}(X, D(\varphi)) = \pi_j$ for all $\pi : i \to j$.

□

Corollary 7.1.4 · *A locally small category $\mathscr{C}$ has all small limits if and only if it has all small products and equalisers (the dual statement, requiring small coproducts and coequalisers, holds for colimits).*

Proof. Small products and equalisers are small limits. We thus need only prove small limits exist if small products and equalisers exist. To this end, let $D : \mathcal{J} \to \mathscr{C}$ be a diagram. Consider the equalizer

$$E = \mathrm{eq}(\prod_{i\in\mathcal{J}} D(i) \rightrightarrows \prod_{\varphi:i\to j} D(j)).$$

By Corollary 7.1.3, the set of morphisms into E can be computed as a limit in set:

$$\mathrm{Hom}_{\mathscr{C}}(X, E) \cong \mathrm{eq}(\mathrm{Hom}_{\mathscr{C}}(X, \prod_{i\in\mathcal{J}} D(i)) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(X, \prod_{\varphi:i\to j} D(j))).$$

Applying Corollary 7.1.3 again to the objects of this equalizer yields and using Remark 7.1.2 yields

$$\mathrm{Hom}_{\mathscr{C}}(X, E) \cong \mathrm{eq}(\prod_{i\in J} \mathrm{Hom}_{\mathscr{C}}(X, D(i)) \rightrightarrows \prod_{\varphi:i\to j} \mathrm{Hom}_{\mathscr{C}}(X, D(j))) \cong \lim_{i\in\mathcal{J}} \mathrm{Hom}_{\mathscr{C}}(X, D(i)).$$

□

Corollary 7.1.5 · *The colimit of any diagram in* **Set** *exists, and it is given by*

$$\mathrm{coeq}(\coprod_{\varphi:i\to j} D(i) \rightrightarrows \coprod_{i\in j} D(i))$$

Corollary 7.1.6 · *Let $D : \mathcal{J} \to \mathscr{C}$ be a diagram with $\mathcal{J}$ filtered. Then*

$$\underrightarrow{\operatorname{colim}}_{i\in\mathcal{J}} D(i) = (\coprod_{i\in\mathcal{J}} D(i))/_{\sim}$$

where $a \sim b$ for $a \in D(i)$ and $b \in D(j)$ if and only if there exist maps $\varphi : i \to k$ and $\psi : j \to k$ such that $D(\varphi)(a) = D(\psi)(b)$.

proof of this, some prose.

7.2 Limits and colimits of abelian groups

'Bootstrap, bootstrap, bootstrap.'

Lemma 7.2.1 · *Every diagram in* **Ab** *has a limit and a colimit.*

Proof. We saw (co)products and (co)equalisers in the previous lecture. Products of abelian groups are products on the underlying sets with the usual group structure. Coproducts are direct sums. Equalisers are kernels, coequalisers are cokernels:

$$\mathrm{eq}(A \underset{\mathrm{g}}{\overset{\mathrm{f}}{\rightrightarrows}} B) = \ker(f-g)$$
$$\mathrm{coeq}(A \underset{\mathrm{g}}{\overset{\mathrm{f}}{\rightrightarrows}} B) = \mathrm{coker}(f-g).$$

Thus we can use Corollary 7.1.4 to compute limits and colimits from (co)products and (co)equalizers via kernels and cokernels. □

Remark 7.2.2 · If $\mathcal{J}$ is filtered, the colimit is easier: the filtered colimit

$$\underrightarrow{\mathrm{colim}}\, D(i) = (\coprod(i))/\sim$$

in **Set** gets an abelian group structure as follows: if $a \in D(i)$ and $b \in D(j)$, then choose maps $\varphi : i \to k$ and $\psi : j \to k$ (using that $\mathcal{J}$ is filtered) and set

$$a + b = D(\varphi)(a) + D(\psi)(b).$$

One needs to check this is well defined and that it indeed turns the colimit in **Set** to the colimit in **Ab**. This result actually follows from a more general fact: forgetful functors create filtered colimits [Rie16, Theorem 5.6.5]

this reference deals with monadic functors, not immediately obvious it proves our claim. A small discussion of this would be nice.

7.3 Limits and colimits of presheaves

Recall that a presheaf to a category $\mathcal{D}$ is a contravariant functor $F \in \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$. In this course the category $\mathcal{D}$ has been **Set** or **Ab**, for which we know small limits and colimits exist. The following lemma appears as [Rie16, Proposition 3.3.9] stripped of the context of presheaves. It tells us that if $\mathcal{C}$ is small, the work we've done so far for **Set** and **Ab** suffices to show presheaves into these categories have small limits and colimits. Furthermore, we can compute (co)limits of such presheaves objectwise. We have seen these claims before in Lemma 2.1.3.

Lemma 7.3.1 · *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be a category with small limits and colimits. Then any diagram $D : \mathcal{J} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ has a limit and a colimit, computed objectwise:*

$$(\lim_{i\in\mathcal{J}} D(i))(U) = \lim_{i\in\mathcal{J}}(D(i)(U))$$
$$(\mathrm{colim}_{i\in\mathcal{J}} D(i))(U) = \mathrm{colim}_{i\in\mathcal{J}}(D(i)(U))$$

for all $U \in \mathcal{C}$.

Proof. We do limits; colimits follow dually. We will turn $U \mapsto \lim_{i\in\mathcal{J}}(D(i)(U))$ into a functor $\mathcal{C}^o p \to \mathcal{D}$. Given $f : U \to V$ in $\mathcal{C}$, we get morphisms

$$\lim_{i\in\mathcal{J}} D(i)(V) \xrightarrow{\pi_i} D(i)(V) \xrightarrow{D(i)(f)} D(i)(U)$$

turning $\lim_{i \in \mathcal{J}} D(i)(V)$ into a cone over $D(-)(U)$ by naturality of $D(-)(f)$. This gives a unique map

$$\lim_{i \in \mathcal{J}} D(i)(V) \to \lim_{i \in \mathcal{J}} D(i)(U)$$

which turns $U \mapsto \lim_{i \in J} D(i)(U)$ into a functor. It is then clearly a limit of D. For more details, see [Mac71, Chapter 5.3] □

Corollary 7.3.2 · *The presheaf categories* **PSh**(X) *and* **PAb**(X) *have all small limits and colimits.*

7.4 Limits and colimits of sheaves

Theorem 7.4.1 · *Any diagram $D : \mathcal{J} \to$ **Sh**(X) (or $D : \mathcal{J} \to$ **Ab**(X)) has a limit, computed in* **PSh**(X) *(***PAb**(X)*). Such a diagram also has a colimit in* **Sh**(X) *(***Ab**(X)*), obtained by sheafification of the colimit in* **PSh**(X) *(***PAb**(X)*).*

Proof. For limits, it suffices to show that the limit in presheaves is a sheaf: we saw that limits in **PSh**(X) are computed objectwise. If $U = \cup_{i \in I} U_i$ is an open cover, then

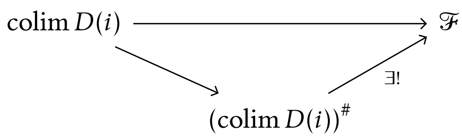
$$\lim_{k \in \mathcal{J}} D(k)(U) \to \prod_{i \in I} \lim_{k \in \mathcal{J}} D(k)(U_i) \rightrightarrows \prod_{i,j \in I} \lim_{k \in \mathcal{J}} D(k)(U_i \cap U_j)$$

is an equaliser: either check by hand, or use that limits commute with limits.

For colimits, we get a cocone $(D(j) \to (\operatorname{colim} D(i))^{\#})_{j \in \mathcal{J}}$. Given any other cocone $(D(j) \to \mathcal{F})_{j \in \mathcal{J}}$, we get a unique morphism of co-cones in presheaves:

$$(D(j) \to \operatorname{colim}_{i \in \mathcal{J}} D(i))_{j \in \mathcal{J}} \to (D(j) \to \mathcal{F})_{j \in \mathcal{J}}.$$

The universal property of sheafification gives a unique factorisation



which shows that the sheafified presheaf is the colimit. □

Example 7.4.2 · The terminal object in **Sh**(X) is the constant sheaf $* = h_{X/X}$ given by $U \mapsto *$ for all opens $U \subseteq X$. The coproduct $\coprod_{s \in S} *$ in **PSh**(X) is the constant presheaf $U \mapsto S$, whose sheafification is the constant sheaf $\underline{S}$. Thus $\underline{S} \cong \coprod_{s \in S} *$ in **Sh**(X), which is also clear in **LocalHomeo**$/X$.

Give more detail in this and the last proof.

LECTURE 8

Additive and abelian categories

We will set sheaves aside for the moment and layout the algebraic foundations for the lectures to come, in which we will study the field of *homological algebra*. In this lecture, we define *abelian categories*. The idea is to axiomatise the defining properties of the category of modules over a ring. As a result, categories of R-modules for some ring R will be abelian. Strikingly, this axiomatisation is 'strict' in the sense of the Freyd–Mitchell theorem (which will not be treated in this course), which says that any small abelian category can be embedded in some category of modules [Wei94, Theorem 1.6.1].

8.1 Additive categories

'We can only write down diagrams,
prove they commute, and run away
as fast as we can.'

Definition 8.1.1 · A *pre-additive* category $\mathscr{C}$ is a locally small category where each hom-set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is endowed with the structure of an abelian group such that composition

$$\mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z), \quad (g, f) \mapsto gf$$

is bilinear.

Expanding the definitions, this means in particular that between every two objects X and Y in a pre-additive category, there is a zero map $0 : X \to Y$, and that two parallel maps $f, g : X \to Y$ have a sum $f + g : X \to Y$.

The following lemma shows that in a pre-additive category, finite coproducts and finite products coincide.

Lemma 8.1.2 · *Let $\mathscr{C}$ be a pre-additive category.*

(i) If $X \in \mathscr{C}$, then the following are equivalent:

(a) X is initial;

(b) X is terminal;

(c) $\mathrm{id}_X = 0 \in \mathrm{Hom}_{\mathscr{C}}(X, X)$;

(ii) If $X, Y, Z \in \mathscr{C}$ then the following are equivalent:

(a) There are maps

$$X \xrightarrow{i} Z \xleftarrow{j} Y$$

making Z the coproduct.

(b) *There are maps*

$$Z \xrightarrow{p} X \xleftarrow{q} Y$$

making Z the product.

(c) *There are maps*

$$X \underset{p}{\overset{i}{\rightleftarrows}} Z \underset{q}{\overset{j}{\leftrightarrows}} Y$$

such that $p \circ i = \mathrm{id}_X$ *and* $q \circ j = \mathrm{id}_Y$ *and* $i \circ p + j \circ q = \mathrm{id}_Z$.

Definition 8.1.3 · A *biproduct* of two objects X and Y in a pre-additive category is an object Z with maps

$$X \underset{p}{\overset{i}{\rightleftarrows}} Z \underset{q}{\overset{j}{\leftrightarrows}} Y$$

satisfying $pi = \mathrm{id}_X$, $qj = \mathrm{id}_Y$ and $ip + jq = \mathrm{id}_Z$.

Remark 8.1.4 · Note that these force $q \circ i$ and $p \circ j$ to be zero. To see this it suffices to show that $j \circ q \circ i = 0$, because j is monic. This holds because $jqi = (\mathrm{id}_Z - ip)i = i - ipi = i - i = 0$.

Proof (of Lemma 8.1.2). (i) If X is initial, then $\mathrm{Hom}_{\mathscr{C}}(X, X) = 0$ so $\mathrm{id}_X = 0$. Conversely, if $\mathrm{id}_x = 0$ then every map $f : X \to Y$ i n $\mathscr{C}$ satisfies $f = f \circ \mathrm{id}_X = f \circ 0 = 0$. So X is initial. This proves (a) is equivalent to (b), and it follows dually that (b) is equivalent to (c).

(ii) If

$$X \underset{p}{\overset{i}{\rightleftarrows}} Z \underset{q}{\overset{j}{\leftrightarrows}} Y$$

is a biproduct, we saw $qi = 0$ and $pj = 0$. Then $X \xrightarrow{i} Z$ and $Y \xrightarrow{j} Z$ is a coproduct. If

$$X \xrightarrow{f} W \xleftarrow{g} Y$$

is any cocone, set $h = fp + gq : Z \to W$. Then $hi = (fp + gq)i = fpi + gqi = f \circ \mathrm{id}_X + g \circ 0 = f$, and likewise $hj = g$. If $h' : Z \to W$ satisfies $h'i = f$ and $h'j = g$ then $h' = h' \circ \mathrm{id}_Z = h' \circ (ip + jq) = h'ip + h'jq = fp + gq = h$. So

$$X \xrightarrow{i} Z \xleftarrow{j} Y$$

form a coproduct. Conversely, if these maps form a coproduct, then the cocones

$$X \xrightarrow{\mathrm{id}} X \xleftarrow{0} Y, \qquad X \xrightarrow{0} Y \xleftarrow{\mathrm{id}} Y$$

define maps $p : Z \to X$, $q : Z \to Y$ with $pi = \mathrm{id}_X, pj = 0, qi = 0, qj = \mathrm{id}_Y$. Then $ip + jq : Z \to Z$ satisfies $(ip + jq)i = ipi + jqi = i \circ \mathrm{id}_X + j \circ 0 = i$, and likewise $(ip + jq) \circ j = \ldots = j$. So $ip + jq = \mathrm{id}_Z$ by the universal property of the coproduct. This proves that (a) is equivalent to (c), and the equivalence of (b) and (c) follows dually. □

Definition 8.1.5 · An *additive* category is a pre-additive category with finite products.

Remark 8.1.6 · By Lemma 8.1.2, if $\mathscr{C}$ is additive then:

- the terminal object is also initial, hence a zero object 0;
- if $X, Y \in \mathscr{C}$ then the biproduct

$$X \underset{p}{\overset{i}{\rightleftarrows}} X \oplus Y \underset{q}{\overset{j}{\leftrightarrows}} Y$$

exists, so $\mathscr{C}$ has finite coproducts;

- if $X_1, \dots, X_n \in \mathscr{C}$ then the map $X_1 \amalg \cdots \amalg X_n \to X_1 \times \cdots \times X_n$ such that the composite

$$X_i \to X_1 \amalg \cdots \amalg X_n \to X_1 \times \cdots \times X_n \to X_j$$

is id_X if $i = j$ and 0 if $i \neq j$ is an isomorphism. We write $X_1 \oplus \cdots \oplus X_n$ for the n-ary biproduct.

Definition 8.1.7 · A *semi-additive* category is a category $\mathscr{C}$ with finite products and coproducts such that the following conditions are satisfied:

(i) the natural map $\varnothing \to *$ is an isomorphism (so $\mathscr{C}$ has a zero object 0),

(ii) for $X_1, \dots, X_n$ in $\mathscr{C}$ the map $X_1 \amalg \cdots \amalg X_n \to X_1 \times \cdots \times X_n$ is an isomorphism.

(Note that we use (i) to define the map in (ii).)

Remark 8.1.8 · While being (pre-)additive is *structure* on a category, being semi-additive is a *property* of a category.

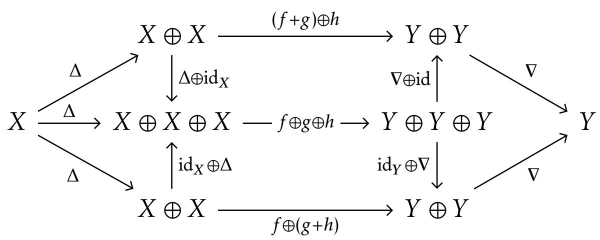
Lemma 8.1.9 · *If $\mathscr{C}$ is semi-additive, then it is canonically enriched in commutative monoids. If $\mathscr{C}$ was additive, then this agrees with the given enrichment in abelian groups (under the inclusion* **Ab** $\hookrightarrow$ **CMon***).*

Proof (sketch, in which the commutativity of the many diagrams is left to the reader). For $f, g : X \to Y$ in $\mathscr{C}$, define $f + g$ to be

$$X \xrightarrow{\Delta} X \times X = X \oplus X \xrightarrow{f \oplus g} Y \oplus Y = Y \amalg Y \xrightarrow{\nabla} Y$$

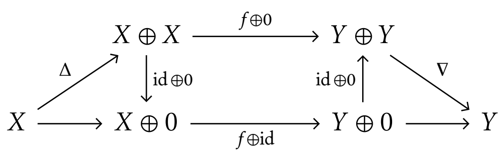
where Δ is the diagonal map and ∇ is the codiagonal.

Associativity: For $f, g, h : X \to Y$, commutativity of



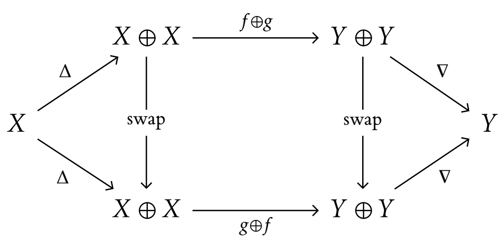
shows that $(f + g) + h = f + (g + h)$.

Unitality: For $f : X \to Y$, commutativity of



shows $f + 0 = f$ and likewise $0 + f = f$.

Commutativity: For $f, g : X \to Y$, commutativity of



gives $f + g = g + f$.

So far, we have proved that $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a commutative monoid.

Distributivity: For $f, f' : X \to Y$ and $g, g' : Y \to Z$, commutativity of

$$\begin{array}{ccccccc} X & \xrightarrow{\Delta} & X \oplus X & \xrightarrow{f \oplus f'} & Y \oplus Y & \xrightarrow{\nabla} & Y \\ & & & \searrow^{gf \oplus gf'} & \downarrow{\scriptstyle g \oplus g} & & \downarrow{\scriptstyle g} \\ & & & & Z \oplus Z & \xrightarrow[\nabla]{} & Z \end{array}$$

shows that $g(f + f') = gf + gf'$ and likewise $(g + g')f = gf + g'f$.

This finishes the proof that $\mathscr{C}$ is enriched in commutative monoids.

Agreement with abelian enrichment: If $\mathscr{C}$ is additive and $f, g : X \to Y$, then let

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} X \oplus X \underset{p_2}{\overset{i_2}{\leftrightarrows}} X$$

be the biproduct, and likewise for Y (by abuse of notation, we use the same names for the injection and projection maps). Then commutativity of

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \oplus X & \xrightarrow{p_2} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \oplus g} & & \downarrow{\scriptstyle g} \\ Y & \xleftarrow[p_1]{} & Y \oplus Y & \xrightarrow[p_2]{} & Y \end{array} \qquad \begin{array}{ccccc} X & \xrightarrow{i_1} & X \oplus X & \xleftarrow{i_2} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \oplus g} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[i_1]{} & Y \oplus Y & \xleftarrow[i_2]{} & Y \end{array}$$

gives $f \oplus g = (f \oplus g) \circ \mathrm{id}_{X \oplus X} = (f \oplus g) \circ (i_1 p_1 + i_2 p_2) = i_1 f p_1 + i_2 g p_2$, and likewise $f \oplus 0 = i_1 f p_1$, $0 \oplus g = i_2 g p_2$, so

$$f \oplus g = (f \oplus 0) + (0 \oplus g).$$

Then bilinearity of composition shows that the composite

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y$$

is $f + g$ (in the enriched sense). □

Corollary 8.1.10 · *Being additive (i.e. admitting an additive structure) is a property.*

Proof. The category $\mathscr{C}$ is additive if and only if if it is semi-additive and all hom-monoids are groups. □

Example 8.1.11 · The category **CMon** of commutative monoids is semi-additive (omitted), but not additive since $\mathrm{Hom}(\mathbb{N}, \mathbb{N}) = \mathbb{N}$ does not have inverses.

8.2 Abelian categories

Exercise 8.2.1 · If $\mathscr{C}$ is pre-additive, show that

$$\mathrm{eq}\big(X \underset{g}{\overset{f}{\rightrightarrows}} Y\big) = \mathrm{eq}\big(X \underset{0}{\overset{f-g}{\rightrightarrows}} Y\big) =: \ker(f - g)$$

(if one of them exists).

So an additive category $\mathscr{C}$ has finite limits or colimits if and only if it has respectively kernels or cokernels.

Definition 8.2.2 · A *pre-abelian* category is an additive category with kernels and cokernels.

Definition 8.2.3 · Let $\mathscr{C}$ be a pre-abelian category and $f : X \to Y$ in $\mathscr{C}$. The *image* of f is $\mathrm{im}(f) := \ker(Y \to \mathrm{coker}(f))$. The *coimage* of f is $\mathrm{coim}(f) := \mathrm{coker}(\ker(f) \to X)$.

Warning: the zoo of names will get worse.

Lemma 8.2.4 · *Any $f : X \to Y$ in a pre-abelian category factors uniquely via*

$$X \to \mathrm{coim}(f) \to \mathrm{im}(f) \to Y.$$

Proof. The composition $\ker(f) \to X \to Y$ is 0. So there is a unique factorisation

$$X \to \mathrm{coim}(f) \to Y$$

of f. The composition $\mathrm{coim}(f) \to Y \to \mathrm{coker}(f)$ is 0 (since this holds after precomposition with $X \to \mathrm{coim}(f)$). So there is a unique factorisation

$$\mathrm{coim}(f) \to \mathrm{im}(f) \to Y. \qquad \square$$

And now the moment we have all been waiting for: the definition of an abelian category.

Definition 8.2.5 · An *abelian* category is a pre-abelian category $\mathscr{A}$ such that the induced map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism for all $f : X \to Y$ in $\mathscr{A}$.

Example 8.2.6 · The category **Ab** of abelian groups and more generally the category $\mathbf{Mod}_R$ of (left or right) modules over a ring R are abelian categories, and the coimage–image isomorphims in these categories is the first isomorphism theorem: $X/\ker(f) \cong \mathrm{im}(f)$.

Correspondingly, we also call the condition of Definition 8.2.5 'the first isomorphism theorem' in general pre-abelian categories.

The following lemma tells us that the category of presheaves of abelian groups on a topological space is an abelian category.

Lemma 8.2.7 · *If $\mathscr{A}$ is abelian and $\mathscr{C}$ is small, then $\mathbf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{A})$ is abelian. In particular, the category $\mathbf{PAb}(X)$ of abelian presheaves on a space X is abelian.*

Proof. If $\alpha, \beta : F \implies G$ is a natural transformation, then define the sum $\alpha + \beta : F \implies G$ by $(\alpha + \beta)_U = \alpha_U + \beta_U$. This is natural since the maps $F(V) \to F(U)$ and $G(V) \to G(U)$ are group homorphisms for any map $U \to V$ in $\mathscr{C}$. This turns $\mathbf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{A})$ into a pre-additive category. All the other questions (existence of finite, limits and colimits, the first isomorphism theorem) are checked objectwise. $\square$

To summarise the various notions we introduced in this lecture:

- A *pre-additive* category is a category enriched in $(\mathbf{Ab}, \otimes)$.
- An *additive* category is a pre-additive category with finite products (and then it automatically has finite coproducts which coincide with finite products).
- A *pre-abelian* category is an additive category with kernels and cokernels (so a pre-additive category with finite limits and colimits).
- An *abelian* category is a pre-abelian category in which the first isomorphism theorem holds, that is, the natural map $\mathrm{coim}\, f \to \mathrm{im}\, f$ is an isomorphism for every map f. Spelling out everything, this means that it is enriched in $(\mathbf{Ab}, \otimes)$, has finite limits and colimits, and that the first isomorphism theorem holds.

LECTURE 9

Monomorphisms and epimorphisms, chain complexes, exact sequences

In the last lecture, we defined abelian categories as **Ab**-enriched categories $\mathscr{A}$ with finite limits and colimits in which the first isomorphism theorem holds (the natural map $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism for all $f : A \to B$ in $\mathscr{A}$). We also showed that the category $\mathbf{PAb}(X)$ of abelian presheaves on a space X is an abelian category.

Today we will show that also the category $\mathbf{Ab}(X)$ of abelian sheaves on X is an abelian category, and we will give a description of monomorphisms and epimorphisms.

9.1 Abelian category of abelian sheaves

Lemma 9.1.1 · *Let $f : Y \to X$ be a continuous map. Then the pullback functors $f^* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and $f^* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$ preserve finite limits and all colimits.*

Proof. Since the pullback functor f^* is left adjoint to the pushforward functor f_* (Proposition 5.2.5), it preserves colimits.

For finite limits, there are two methods:

(i) Use that f^* is given by

$$\mathbf{LocalHomeo}_{/X} \to \mathbf{LocalHomeo}_{/Y}, \quad (Z \to X) \mapsto \big(Z \underset{X}{\times} Y \to Y\big),$$

which preserves finite limits, and finite limits in $\mathbf{LocalHomeo}_{/-}$ are computed as in $\mathbf{Top}_{/-}$.[1]

(ii) Use that

$$(f^{\circledast}\mathscr{F})(U) = \operatorname*{colim}_{f(U)\subseteq V} \mathscr{F}(V)$$

is a filtered colimit, and filtered colimits in **Set** and **Ab** commute with finite limits (Additional exercise 9.1). Check that sheafification preserves finite limits (Homework 5). □

Proposition 9.1.2 · *If X is a topological space, then the category $\mathbf{Ab}(X)$ of abelian sheaves on X is an abelian category.*

Proof. Being a full subcategory of the abelian category $\mathbf{PAb}(X)$, the category $\mathbf{Ab}(X)$ is pre-additive, and we have already shown in Theorem 7.4.1 that it has all limits and colimits.

It remains to check that the first isomorphism theorem holds in $\mathbf{Ab}(X)$. Let $f : \mathscr{F} \Rightarrow \mathscr{G}$ be a map in $\mathbf{Ab}(X)$. We can factor f as

$$\mathscr{F} \Rightarrow \operatorname{coim} f \Rightarrow \operatorname{im} f \Rightarrow \mathscr{G}. \tag{9.1}$$

[1] This does not hold for all limits; it fails for instance for infinite products, already when X is a point.

By Lemma 9.1.1, the formation of the kernel, cokernel, image and coimage commutes with the stalk functor i_x^* for all $i_x : \{x\} \to X$ (see Definition 5.3.1). So (9.1) induces the factorisation

$$\mathcal{F}_x \to (\operatorname{coim} f)_x = \operatorname{coim} f_x \to (\operatorname{im} f)_x = \operatorname{im} f_x \to \mathcal{G}_x$$

of the map f_x induced on stalks. The map $\operatorname{coim} f_x \to \operatorname{im} f_x$ is an isomorphism since **Ab** is an abelian category. Since this holds for all stalks, the map $\operatorname{coim} f \to \operatorname{im} f$ is also an isomorphism by Lemma 5.3.5. □

9.2 Monomorphisms and epimorphisms

Lemma 9.2.1 · *Let $\mathcal{A}$ be an abelian category and $f : A \to B$ a map in $\mathcal{A}$. Then:*

(i) f is monic if and only if $\ker f = 0$*;*

(ii) f is epic if and only if $\operatorname{coker} f = 0$.

The recipe of the proof is: Yoneda + the same statement in **Ab**.

Proof. (i) By definition, $\ker f$ represents $\ker(\operatorname{Hom}_{\mathcal{A}}(-, A) \Rightarrow \operatorname{Hom}_{\mathcal{A}}(-, B))$. So $\ker f = 0$ if and only if $\operatorname{Hom}_{\mathcal{A}}(-, A) \Rightarrow \operatorname{Hom}_{\mathcal{A}}(-, B)$ is an injective map of presheaves, that is, $f : A \to B$ is monic.

(ii) Follows dually. □

For sheaves, let's make it quite concrete.

Lemma 9.2.2 · *Let $f : \mathcal{F} \Rightarrow \mathcal{G}$ be a map in* $\mathbf{Sh}(X)$ *(resp.* $\mathbf{Ab}(X)$*). Then the following are equivalent:*

(i) f is monic (in $\mathbf{Sh}(X)$ *resp.* $\mathbf{Ab}(X)$*);*

(ii) f is monic in $\mathbf{PSh}(X)$ *(resp.* $\mathbf{PAb}(X)$*);*

(iii) f_U is injective for all open subsets $U \subseteq X$;

(iv) f_x is injective for all points $x \in X$;

(v) $\operatorname{sp}(f) \to X$ *is injective.*

Proof. By Additional exercise 7.3, we know that f is monic if and only if the diagram

$$\begin{array}{ccc} \mathcal{F} & \xRightarrow{\text{id}} & \mathcal{F} \\ {\scriptstyle \text{id}}\Downarrow & & \Downarrow{\scriptstyle f} \\ \mathcal{F} & \xRightarrow[f]{} & \mathcal{G} \end{array}$$

is a pullback square. So the equivalence of (i) and (ii) follows since the inclusion $\mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$ (resp. $\mathbf{Ab}(X) \hookrightarrow \mathbf{PAb}(X)$) creates limits (Theorem 7.4.1), and the equivalence of (ii) and (iii) since limits in $\mathbf{PSh}(X)$ (resp. $\mathbf{PAb}(X)$) are computed objectwise.

The functors

$$\mathbf{Sh}(X) \xrightarrow{\simeq} \mathbf{LocalHomeo}_{/X} \hookrightarrow \mathbf{Top}_{/X} \hookrightarrow \mathbf{Set}_{/X}$$

(and the corresponding functors for abelian sheaves) create fibre products, so they preserve and reflect monomorphisms. Since monomorphisms in $\mathbf{Set}_{/X}$ (resp. $\mathbf{Ab}(\mathbf{Set}_{/X})$) are injective maps, this proves that (i) is equivalent to (v).

Finally, (iv) is equivalent to (v) since the fibres of $\operatorname{sp}(\mathcal{F}) \to \operatorname{sp}(\mathcal{G})$ over $x \in X$ are the stalks f_x. □

The epimorphisms of sheaves are more subtle. Although the epimorphisms of presheaves are just objectwise epimorphisms (characterisation (iii) of the last lemma), this is not true for epimorphisms of sheaves. This is perhaps to be expected: monomorphisms are related to limits (as discussed in the last proof), and the forgetful functor from sheaves to presheaves creates limits, but epimorphisms are related to colimits (in a dual way, made precise in the following proof), whose relation to colimits of presheaves is not so straightforward.

Lemma 9.2.3 · *Let $f : \mathscr{F} \Rightarrow \mathscr{G}$ be a map in* $\mathbf{Sh}(X)$ *(resp.* $\mathbf{Ab}(X)$*). Then the following are equivalent:*

(i) f is epic (in $\mathbf{Sh}(X)$ resp. $\mathbf{Ab}(X)$);

(ii) for every open subset $U \subseteq X$ and every section $t \in \mathscr{G}(U)$, there exists an open cover $U = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathscr{F}(U_i)$ such that $f(s_i) = t|_{U_i}$;

(iii) f_x is surjective for all points $x \in X$;

(iv) $\mathrm{sp}(f) \to X$ is surjective.

In $\mathbf{Ab}(X)$, these statements are also equivalent to:

(v) the sheafification of the presheaf cokernel is zero.

Proof. Again, f is epic if and only if the diagram

$$\begin{array}{ccc} \mathscr{F} & \overset{f}{\Longrightarrow} & \mathscr{G} \\ {\scriptstyle f}\Downarrow & & \Downarrow{\scriptstyle \mathrm{id}} \\ \mathscr{G} & \underset{\mathrm{id}}{\Longrightarrow} & \mathscr{G} \end{array}$$

is a pushout square. If this holds, it holds for all stalks $f_x : \mathscr{F}_x \to \mathscr{G}_x$ since i_x^* preserves colimits by Lemma 9.1.1. This proves that (i) implies (iii), and the equivalence of (iii) and (v) is clear since surjectivity in $\mathbf{Top}_{/X}$ is checked fibrewise. Conversely, that (v) implies (i) follows from the equivalence $\mathbf{Sh}(X) \simeq \mathbf{LocalHomeo}_{/X}$ (resp. $\mathbf{Ab}(X) \simeq \mathbf{Ab}(\mathbf{LocalHomeo}_{/X})$) since a surjection in $\mathbf{LocalHomeo}_{/X}$ is surely an epimorphism[2].

The equivalence of (ii) and (iii) is an unwinding of the definitions. Statement (ii) means that for all $x \in U$ and all $t \in \mathscr{G}(U)$ there exists $x \in U' \subseteq U$ and $s \in \mathscr{F}(U)$ such that $f(s) = t|_{U'}$, which is (iii).

In the abelian case, the equivalence of (i) and (v) follows from the construction of the sheaf cokernel as the sheafification of the presheaf cokernel. □

Definition 9.2.4 · A map in an abelian category is called *injective* if it is a monomorphism and *surjective* if it is an epimorphism.

warning sign?

Remark 9.2.5 · If $f : \mathscr{F} \Rightarrow \mathscr{G}$ is a surjective map of abelian sheaves, then the component $\mathscr{F}(U) \to \mathscr{G}(U)$ need not be surjective.

In Homework 3, we constructed a surjection $f : \mathscr{F} \Rightarrow i_* i^* \mathscr{F}$ for any closed inclusion $i : Z \hookrightarrow X$ and any abelian sheaf $\mathscr{F}$ on X. Take $X = \mathbb{R}$, $Z = \{0, 1\} \subseteq \mathbb{R}$ and $F = \underline{\mathbb{Z}}$. Then $i^* \mathscr{F} = \underline{\mathbb{Z}}$ (pullbacks of constant sheaves are constant sheaves) and $\underline{\mathbb{Z}} \to i_* \underline{\mathbb{Z}}$ is surjective. But plugging in $\mathbb{R}$ gives the map

$$\underline{\mathbb{Z}}(\mathbb{R}) = \mathbb{Z} \to (i_* \underline{\mathbb{Z}})(\mathbb{R}) = \underline{\mathbb{Z}}(\mathbb{R} \cap \{0, 1\}) = \mathbb{Z} \oplus \mathbb{Z}, \quad a \mapsto (a, a).$$

which is not surjective. However, any $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ can be lifted: the open sets $U_0 := (-\infty, 1)$, $U_1 := (0, \infty)$ cover $\mathbb{R}$ and (a, b) locally lifts to $s_0 := a$ and $s_1 := b$, but they do not glue. (Another example is on Homework 5.)

Picture

[2] This holds in any *concrete category* $\mathscr{C}$, a category with a faithful functor $\mathscr{C} \to \mathbf{Set}$, since faithful functors reflect epimorphisms.

9.3 Exact sequences

Lemma 9.3.1 · *Let $\mathscr{A}$ be an abelian category and $f : A \to B$ a map in $\mathscr{A}$. Then:*

(i) The map $\ker f \to A$ *(resp.* $B \to \operatorname{coker} f$*) is monic (resp. epic), and an isomorphism if* $f = 0$.

(ii) The map $\operatorname{im} f \to B$ *(resp.* $A \to \operatorname{coim} f$*) is monic (resp. epic), and an isomorphism if f is epic (resp. monic).*

(iii) $\ker f$ *is canonically isomorphic to* $\ker(A \to \operatorname{coim} f)$*, and* $\operatorname{coker} f$ *is canonically isomorphic to* $\operatorname{coker}(\operatorname{im} f \to B)$.

(iv) If f is monic (resp. epic), then $A \cong \ker(B \to \operatorname{coker} f)$ *(resp.* $B \cong \operatorname{coker}(\ker f \to A)$*). (In particular, any monomorphism (resp. epimorphism) is an equaliser (resp. coequaliser), hence a* regular *monomorphism (resp. epimorphism).)*

(v) If f is both monic and epic, then it is an isomorphism. ($\mathscr{A}$ is a balanced *category.)*

Proof. (i) The universal property of $\ker f$ is

$$\operatorname{Hom}_{\mathscr{A}}(C, \ker f) \cong \{\varphi \in \operatorname{Hom}_{\mathscr{A}}(C, A) \mid f\varphi = 0\},$$

so $\operatorname{Hom}_{\mathscr{A}}(C, \ker f) \to \operatorname{Hom}_{\mathscr{A}}(C, A)$ is injective for all C, hence $\ker f \to A$ is monic. If $f = 0$, the condition $f\varphi = 0$ on φ is vacuous, so $\operatorname{Hom}_{\mathscr{A}}(C, \ker f) \cong \operatorname{Hom}_{\mathscr{A}}(C, A)$. The statements about $B \to \operatorname{coker} f$ follow dually.

(ii) Apply (i) to $B \to \operatorname{coker} f$ (resp. $\ker f \to A$), which we saw is zero if f is epic (resp. monic).

(iii) We have a factorisation

$$A \xrightarrow{\pi} \operatorname{coim} f \xrightarrow{\cong} \operatorname{im} f \rightarrowtail B$$

of f by (ii), so the map $\operatorname{coim} f \to B$ is monic. Then for any $\varphi : C \to A$, the composition $\pi\varphi = 0$ if and only if $f\varphi = 0$, so $\ker \pi = \ker f$. The other statement follows dually.

(iv) If f is monic, then $A \cong \operatorname{coim} f \cong \operatorname{im} f$ by (ii). The other statement follows dually.

(v) By (ii), all maps

$$A \longrightarrow \operatorname{coim} f \xrightarrow{\cong} \operatorname{im} f \longrightarrow B$$

are isomorphisms. □

Definition 9.3.2 · A *cochain complex* in an additive category $\mathscr{C}$ is a diagram

$$\dots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \dots$$

in $\mathscr{C}$ such that $d^{i+1}d^i = 0$ for all $i \in \mathbb{Z}$. (If it is only defined on a subset $I \subseteq \mathbb{Z}$, we set $C^i := 0$ for $i \notin I$.)

A cochain complex $C^\bullet$ is *exact* at C^i if the canonical map $\operatorname{im} d^{i-1} \to \ker d^i$ is an isomorphism.

A *short exact sequence* is an exact complex $0 \to A \to B \to C \to 0$.

A *cochain map* $f^\bullet : C^\bullet \to D^\bullet$ of cochain complexes is a natural transformation in $\mathbf{Fun}(\mathbb{Z}, \mathscr{C})$:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & C^{i-1} & \xrightarrow{d^{i-1}} & C^i & \xrightarrow{d^i} & C^{i+1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\dots & \longrightarrow & D^{i-1} & \xrightarrow[d^{i-1}]{} & D^i & \xrightarrow[d^i]{} & D^{i+1} & \longrightarrow & \dots
\end{array}$$

The following lemma shows that our previous *ad hoc* notion of exactness of Definition 6.2.6 agrees with the categorical notion.

Lemma 9.3.3 · *Let $\mathcal{F}^\bullet$ be a cochain complex in* $\mathbf{Ab}(X)$*. Then $\mathcal{F}^\bullet$ is exact at $\mathcal{F}^i$ if and only if $\mathcal{F}^\bullet_x$ is exact at $\mathcal{F}^i_x$ for all $x \in X$.*

Proof. We saw in Lemma 9.1.1 that $i_x^* : \mathbf{Ab}(X) \to \mathbf{Ab}$ preserves all finite limits and colimits, so in particular the kernel, cokernel, image and coimage. Thus, we conclude since $\operatorname{im} d^{i-1} \Rightarrow \ker d^i$ is an isomorphism if and only if $(\operatorname{im} d^{i-1})_x \to (\ker d^i)_x$ is an isomorphism for all $x \in X$ by Lemma 5.3.5. □

LECTURE 10

Exact functors, diagram lemmas

10.1 Exact functors

> 'For the five lemma, if you look at three different sources, you'll get three different lemmas. That's the three lemma.'

Definition 10.1.1 · Let $\mathscr{C}$ and $\mathscr{D}$ be categories and assume $\mathscr{C}$ has finite limits (resp. finite colimits). Then a functor $F : \mathscr{C} \to \mathscr{D}$ is *left exact* (resp. *right exact*) if it preserves finite limits (resp. finite colimits). The functor F is *exact* if it is both left and right exact.

The goal of today is to show that this definition agrees with the definition in terms of short exact sequences.

Example 10.1.2 ·

- If F has a right adjoint, then it is right exact. If F *is* a right adjoint (so *has* a left adjoint), then it is left exact.
- We proved that $f^* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ for $f : Y \to X$ is exact in Lemma 9.1.1.
- On Homework 5, we show that sheafification $(-)^\sharp : \mathbf{PSh}(X) \to \mathbf{Sh}(X)$ (or for abelian sheaves) is exact.
- The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is exact since it has adjoints on both sides, equipping a set with the discrete or indiscrete topology.
- The forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is left exact, but not right exact since $A \oplus B \neq A \amalg B$.

Definition 10.1.3 · Let $\mathscr{C}$ and $\mathscr{D}$ be pre-additive categories. Then a functor $F : \mathscr{C} \to \mathscr{D}$ is *additive* if the maps $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(FX, FY)$ are group homomorphisms.

Most functors between abelian categories in nature are additive.

Example 10.1.4 ·

- If $\mathscr{C}$ is a pre-additive category with an object X, then the representable functor $\mathrm{Hom}_{\mathscr{C}}(-, X) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Ab}$ is additive: the map

  $$\mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{Hom}_{\mathscr{C}}(Z, X), \mathrm{Hom}_{\mathscr{C}}(Y, X)), \quad f \mapsto (g \mapsto g \circ f)$$

  is a group homomorphism since composition in a pre-additive category is bilinear. Likewise, the corepresentable functor $\mathrm{Hom}_{\mathscr{C}}(X, -) : \mathscr{C} \to \mathbf{Ab}$ is also additive.
- The free–forgetful adjunction between abelian groups and sets gives a monad (in particular an endofunctor) on $\mathbf{Ab}$, which is not additive: it sends the zero object to $\mathbb{Z}$ and the identity of 0, which is also the zero map, must be sent to the identity of $\mathbb{Z}$ by functoriality and to the zero map of $\mathbb{Z}$ by additivity.

The problem in the last non-example was that the functor did not preserve the zero object.

Lemma 10.1.5 · *Let $\mathscr{C}$ and $\mathscr{D}$ be additive categories and $F : \mathscr{C} \to \mathscr{D}$ a functor. Then the following are equivalent:*

(i) F is additive;

(ii) F preserves finite products;

(iii) F preserves finite coproducts;

(iv) F preserves binary biproducts.

Proof. Note that (iv) implies that F preserves the zero object: the biproduct of the zero object and itself is sent to

$$0 \underset{p}{\overset{i}{\rightleftarrows}} 0 \oplus 0 \underset{q}{\overset{j}{\leftrightarrows}} 0 \qquad \overset{F}{\mapsto} \qquad F(0) \underset{F(p)}{\overset{F(i)}{\rightleftarrows}} F(0) \oplus F(0) \underset{F(q)}{\overset{F(j)}{\leftrightarrows}} F(0)$$

Then

$$\mathrm{id}_{F(0)} = F(\mathrm{id}_0) = F(0 : F(0) \to F(0)) = F(qi) = F(q)F(i) = 0,$$

so $F(0)$ is a zero object.

Then (ii), (iii) and (iv) are equivalent, since binary products, binary coproducts and binary biproducts agree by Lemma 8.1.2.

Next, assume F is additive, and let

$$X \underset{p}{\overset{i}{\rightleftarrows}} X \oplus Y \underset{q}{\overset{j}{\leftrightarrows}} Y$$

be a biproduct in $\mathscr{C}$, that is, $pi = \mathrm{id}_X$, $qj = \mathrm{id}_Y$ and $ip + jq = \mathrm{id}_{X \oplus Y}$. These equations are preserved by F, showing that (i) implies (iv).

Conversely, assume F preserves binary biproducts, and let $f, g : X \to Y$ be maps in $\mathscr{C}$. We saw in Lemma 8.1.9 that $f + g$ is the composition

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$

Applying F gives the diagram

$$F(X) \xrightarrow{\Delta} F(X) \oplus F(X) \xrightarrow{F(f) \oplus F(g)} F(Y) \oplus F(Y) \xrightarrow{\nabla} F(Y).$$

using the universal property of the product to show that $F(\Delta) = \Delta$, the universal property of the coproduct to show that $F(\nabla) = \nabla$, and the either universal property to show that $F(f \oplus g) = F(f) \oplus F(g)$ since we already saw that preserving finite biproducts is equivalent to preserving finite products and finite coproducts. This shows that $F(f + g) = F(f) + F(g)$, showing that (iv) implies (i). □

Corollary 10.1.6 · *Any left exact or right exact functor between pre-abelian categories is additive.*

Lemma 10.1.7 · *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $F : \mathscr{A} \to \mathscr{B}$ a functor. Then:*

(i) F is left exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact in $\mathscr{B}$.

(ii) F is right exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact in $\mathscr{B}$.

(iii) F is exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact in $\mathscr{B}$.

Remark 10.1.8 · For any biproduct

$$A \underset{p}{\overset{i}{\rightleftarrows}} A \oplus B \underset{q}{\overset{j}{\leftrightarrows}} B$$

the sequence $0 \to A \to A \oplus B \to B \to 0$ is exact:

- the inclusion $i : A \to A \oplus B$ is a (split) monomorphism;
- the projection $q : A \oplus B \to B$ is a (split) epimorphism;
- if $f : C \to A \oplus B$ is a map such that the composite $qf : C \to A \oplus B \to B$ is zero, then

  $$f = (ip + jq)f = ipf + jqf = ipf,$$

  so f factors (uniquely) through i.

Proof (of Lemma 10.1.7). (i) Note: if F takes short exact sequences to left exact sequences, then it is additive: we apply the five lemma to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(A) & \xrightarrow{\begin{pmatrix}\mathrm{id}\\0\end{pmatrix}} & F(A) \oplus F(B) & \xrightarrow{(0,\mathrm{id})} & F(B) & \longrightarrow & 0 \\
 & & \Vert & & \big\downarrow{\scriptstyle (F(i),F(j))} & & \Vert & & \big\downarrow \\
0 & \longrightarrow & F(A) & \xrightarrow[F(i)]{} & F(A \oplus B) & \xrightarrow[F(q)]{} & F(B) & \longrightarrow & \mathrm{coker}(F(q))
\end{array}$$

with exact rows. The five lemma shows that F preserves binary coproducts, so F is additive. The result now follows since F preserves finite limits if and only if it preserves finite products and kernels, and $\ker(f : B \to C) = \ker(B \twoheadrightarrow \operatorname{im} f)$ so left exactness is equivalent to preserving kernels.

(ii) Dually.

(iii) From the previous two items. □

10.2 Diagram lemmas

> 'Let me call this proof a sketch, so I can bail out.'

The exposition here follows [Ive86]. The following lemma is a variant of the four lemma, a bit more general than what is usually called the four lemma:

Lemma 10.2.1 (four lemma) · *Let $\mathcal{A}$ be an abelian category and let*

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\
{\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle d} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' \\
\big\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

be a commutative diagram with exact rows and columns. Then the sequence $\ker b \to \ker c \to \ker d$ *is exact.*

Corollary 10.2.2 (five lemma) · *Let $\mathcal{A}$ be an abelian category and let*

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

be a commutative diagram with exact rows. If a is epic, e is monic and b and d are isomorphisms, then c is an isomorphism.

Proof. Apply the four lemma 10.2.1 to a, b, c and d to get an exact sequence

$$0 = \ker b \to \ker c \to \ker d = 0,$$

so $\ker c = 0$. Dually, we see $\operatorname{coker} c = 0$, so c is an isomorphism by Lemma 9.3.1. □

Proof (of Lemma 10.2.1, sketch). Let E and E' be the images of respectively $B \to C$ and $B' \to C'$. Their uiversal properties give commutative diagrams

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & E & \longrightarrow & 0 \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle e} & & \\
A' & \longrightarrow & B' & \longrightarrow & E' & \longrightarrow & 0 \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}
\qquad
\begin{array}{ccccccc}
0 & \longrightarrow & E & \longrightarrow & C & \longrightarrow & D \\
 & & {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\
0 & \longrightarrow & E' & \longrightarrow & C' & \longrightarrow & D'
\end{array}$$

with exact rows and columns. We will check that the sequences

$$0 \to \ker e \to \ker c \to \ker d$$

and

$$\ker b \to \ker e \to 0$$

are exact. Then $\ker b \to \ker c$ has image $\ker e = \ker(\ker c \to \ker d)$.

For the former, suppose we have a map f making the diagram:

$$\begin{array}{ccccccc}
 & & & & X & \overset{0}{\searrow} & \\
 & & & & {\scriptstyle f}\downarrow & & \\
 & & \ker e & \longrightarrow & \ker c & \longrightarrow & \ker d \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E & \longrightarrow & C & \longrightarrow & D \\
 & & {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\
0 & \longrightarrow & E' & \longrightarrow & C' & \longrightarrow & D'
\end{array}$$

commute. Then $X \to D$ is zero, so $X \to C$ factors uniquely through E. Then $X \to E' \to C'$ is zero, so $X \to E'$ is zero, so $X \to E$ factors uniquely through $\ker e$. Dually, the sequence

$$\operatorname{coker} a = 0 \to \operatorname{coker} b \to \operatorname{coker} e \to 0$$

is exact, so $\operatorname{coker} b \to \operatorname{coker} e$ is an isomorphism.

We get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{im}(A' \to B') & \longrightarrow & B' & \longrightarrow & E' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \operatorname{coker} b & \xrightarrow{\cong} & \operatorname{coker} e & \longrightarrow & 0
\end{array}$$

with exact rows, so by the first statement, we get an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im}(A' \to B') & \longrightarrow & \operatorname{im} b & \longrightarrow & \operatorname{im} e & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & A' & & B & \twoheadrightarrow & E & & \end{array}$$

so also an exact sequence

$$A' \to \operatorname{im} b \to \operatorname{im} e \to 0.$$

Now we get

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & E & \longrightarrow & 0 \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle e'} & & \\ A' & \longrightarrow & \operatorname{im} b & \longrightarrow & \operatorname{im} e & \longrightarrow & 0 \end{array}$$

where $\ker b' = \ker b$ and $\ker e' = \ker e$. Then we claim that $\operatorname{im} e = \operatorname{coker}(\ker b' \to E)$: given a map g making the diagram

$$\begin{array}{ccccccc} & & \ker b' & & & \overset{0}{\searrow} & \\ & & \downarrow & & & & \\ A & \longrightarrow & B & \longrightarrow & E & \xrightarrow[g]{} & Y \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle e'} & & \\ A' & \longrightarrow & \operatorname{im} b & \longrightarrow & \operatorname{im} e & \longrightarrow & 0 \end{array}$$

commute, the composite $B \to Y$ factors uniquely through $\operatorname{im} b$. Then $A \to Y$ is zero, so $A' \to Y$ is zero as a is an epimorphism. So $\operatorname{im} b \to Y$ factors uniquely through $\operatorname{im} e$. Then

$$\ker b' \to E \xrightarrow{e'} \operatorname{im} e \to 0$$

is exact, so the map $\ker b' \to \ker e'$ is an epimorphism. □

Lemma 10.2.3 (Snake lemma) · *Given a commutative diagram*

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

in an abelian category, if the rows are exact then there is an exact sequence

$$\ker(a) \to \ker(b) \to \ker(c) \xrightarrow{\partial} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c).$$

The map ∂ is often called the connecting homomorphism*.*

LECTURE 11

Injective and projective objects, resolutions

> ‘08N5 sounds like a bird flu variant – maybe a my generation joke.’

Some more remarks on the four and five lemma which we discussed last week:

- We followed the idea in [Ive86] to check exactness only *once*.
- *Warning*: Checking exactness in an abelian category might be cumbersome. To check $A \to B \to C$ is exact we *cannot* use $\mathrm{Hom}_{\mathscr{A}}(X, -)$ since

$$\mathrm{Hom}_{\mathscr{A}}(X, A) \to \mathrm{Hom}_{\mathscr{A}}(X, B) \to \mathrm{Hom}_{\mathscr{A}}(X, C)$$

  need not be exact.
- An alternative method [Sta24, Tag 08N5]: $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if for any $h : X \to B$ with $g \circ h = 0$, there exists an epimorphism $k : Y \twoheadrightarrow X$ such that $hk : Y \to B$ lifts to A:

$$\begin{array}{ccccc} Y & \overset{k}{\twoheadrightarrow} & X & & \\ \downarrow & & \downarrow & \overset{0}{\searrow} & \\ A & \underset{f}{\longrightarrow} & B & \underset{g}{\longrightarrow} & C \end{array}$$

  (Idea: take $Y \twoheadrightarrow X$ to be $A \times_{\ker g} X$ – see Additional exercise 10.3 for surjectivity of $Y \twoheadrightarrow X$ since $A \twoheadrightarrow \ker g$.) This reduces diagram lemmas to point-set-like statements, but is not easier.

The snake lemma was not proven last week, nor will it be this week, for the proof see [Ive86, § I.1].

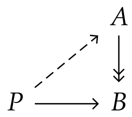
11.1 Injective and projective objects

> ‘We do what we always do in commutative algebra: you Zorn it.’

Definition 11.1.1 · An object X of a category $\mathscr{C}$ is *injective* (resp. *projective*) if $\mathrm{Hom}_{\mathscr{C}}(-, X)$ (resp. $\mathrm{Hom}_{\mathscr{C}}(X, -)$) preserves epimorphisms. Explicitly, given a map $f : A \to X$ and a monomorphism $A \hookrightarrow B$, f extends to B:

$$\begin{array}{ccc} A & \hookrightarrow & B \\ {\scriptstyle f}\downarrow & \overset{\exists}{\swarrow} & \\ X & & \end{array}$$

if X is injective. Dually, X is projective whenever, given an $f : X \to B$ and an epimorphism $A \twoheadrightarrow B$, the map f factors through A:



Recall that in Additional exercise 10.1(h), (i), we showed that the functors

$$\mathrm{Hom}_{\mathscr{A}}(-, X) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}, \quad \mathrm{Hom}_{\mathscr{A}}(X, -) : \mathscr{A} \to \mathbf{Ab}$$

are left exact. Thus, if $\mathscr{C} = \mathscr{A}$ is abelian, one way to characterise injective (resp. projective) objects is via exactness of the corresponding (co)representable functor. Another characterisation of projective objects in abelian categories is the following:

Exercise 11.1.2 · Any short exact sequence $0 \to A \to B \to C \to 0$ splits if and only if C is projective. **reference**

Example 11.1.3 (Projective module is summand of free module) · In ${}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$, the free R-module $R^{\oplus I}$ is projective:

$${}_R\mathrm{Hom}(R^{\oplus I}, M) \cong M^I, \quad \varphi \mapsto (\varphi(e_i))_{i \in I}.$$

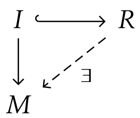
If P is projective, there exists Q (which is automatically projective) such that $P \oplus Q \cong R^{\oplus I}$: choose a surjection $R^{\oplus I} \twoheadrightarrow P$ from a free module (such as $R^{\oplus P} \twoheadrightarrow P$). Since P is projective, by split exactness of the sequence

$$0 \longrightarrow \ker \pi \rightarrowtail R^{\oplus I} \underset{\sigma}{\overset{\pi}{\rightleftarrows}} P \longrightarrow 0,$$

π has a section σ. So $P \oplus Q \cong R^{\oplus I}$ where $Q = \ker \pi = \operatorname{coker} \sigma$.

Example 11.1.4 · Every projective abelian group is free. The idea is to first prove this for finitely generated abelian groups and then by induction using Zorn's lemma. (This probably also holds for R-modules over a PID R.) The key step in the proof is: a subgroup of a free abelian group is free. **source** **cite Lang Appendix 2 section 2**

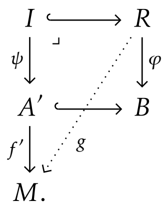
Lemma 11.1.5 · *Let R be a ring and $\mathscr{A} = {}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$. Then $M \in \mathscr{A}$ is injective if and only if for every ideal $I \subseteq R$ every $I \to M$ extends to $R \to M$:*



Proof (Sketch). The forwards implication is immediate. Conversely, suppose the condition holds and let $A \hookrightarrow B$ and $f : A \to M$ in $\mathscr{A}$. Consider the poset,

$$\{(A', f') : A \subseteq A' \subseteq B' \text{ and } f' : A' \to M \text{ extends } f\}$$

ordered by inclusion. This poset satisfies the conditions of Zorn's lemma, so it has a maximal element (A', f'). We claim that $A' = B$. Suppose $A' \neq B$ and choose $x \in B \backslash A'$. Consider $\varphi : R \to B, r \mapsto rx$ and let $I = \varphi^{-1}(A')$. Then we obtain a pullback



where the map g is the unique extension of $f' \circ \varphi$ that we obtain by assumption. But since the sequence

$$0 \longrightarrow I \xrightarrow{\begin{pmatrix}\varphi\\ -\mathrm{id}\end{pmatrix}} A' \oplus R \xrightarrow{(\mathrm{id},\varphi)} A' + Rx \longrightarrow 0$$

is exact, the square

$$\begin{array}{ccc} I & \hookrightarrow & R \\ \downarrow & \ulcorner & \downarrow \\ A' & \longrightarrow & A' + Rx \end{array}$$

is cocartesian. Yet then the morphism $A' + Rx \to M$ induced by f' and g extends f', contradicting maximality. ☐

Exercise 11.1.6 (Homework 6) · An abelian group A is injective if and only if it is *divisible*: the map $A \to A$, $a \mapsto na$ is surjective for all $n \in \mathbb{Z}_{>0}$.

Remark 11.1.7 · If $\mathcal{A}$ is a (locally) presentable abelian category, generated by a set S of objects, then an object X of $\mathcal{A}$ is injective if and only if for every $B \in S$, any extension problem

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & \swarrow & \\ X & & \end{array}$$

has a solution. For example, the categories ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ of left and right R-modules are generated by $S = \{R\}$, and the category $\mathbf{Ab}(X)$ of abelian sheaves on a space X is generated by $\underline{\mathbb{Z}}_U = j_!\underline{\mathbb{Z}}$ for an open subset $j : U \hookrightarrow X$. (But what are the subobjects of $\underline{\mathbb{Z}}_U$?)

Definition 11.1.8 · An abelian sheaf $\mathcal{F}$ on a space X is *flasque* or *flabby* if every diagram

$$\begin{array}{ccc} \underline{\mathbb{Z}}_U & \hookrightarrow & \underline{\mathbb{Z}}_V \\ \downarrow & \swarrow & \\ \mathcal{F} & & \end{array}$$

with $U \subseteq V \subseteq X$ opens has an extension. That is, the restriction $\mathcal{F}(V) \twoheadrightarrow \mathcal{F}(U)$ is surjective for all opens $U \subseteq V \subseteq X$.

Remark 11.1.9 · These names are used differently by some authors e.g. in SGA 4_{II}.

Example 11.1.10 · Injective sheaves are flasque.

11.2 Injective resolutions

> 'This is why you teach courses, to shade on the literature.'

Definition 11.2.1 · An abelian category $\mathcal{A}$ has *enough injectives* (resp. *enough projectives*) if every object X of $\mathcal{A}$ admits a monomorphism $X \hookrightarrow I$ with I injective (resp. an epimorphism $P \twoheadrightarrow X$ with P projective).

Definition 11.2.2 · Let $\mathcal{A}$ be an abelian category and let X be an object of $\mathcal{A}$. An *injective resolution* (resp. *projective resolution*) of X is an exact sequence

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

in which every I^i is injective (resp.

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with all P_i projective).

In particular as we will prove soon, every object admits an injective (resp. projective) resolution in an abelian category with enough injectives (resp. projectives).

Examples 11.2.3 · The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is a projective resolution of the abelian group $\mathbb{Z}/n\mathbb{Z}$.

The short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is an injective resolution of the abelian group $\mathbb{Z}$.

Lemma 11.2.4 · *If $\mathcal{A}$ has enough injective (resp. projectives) then every object X of $\mathcal{A}$ has an injective (resp. projective) resolution.*

Proof. We treat the injective case, the projective case follows dually. Since $\mathcal{A}$ has enough injectives, we can find a monomorphism $X \to I^0$ into an injective object, that is, making the sequence

$$0 \longrightarrow X \longrightarrow I^0$$

is exact, covering the base case. In the inductive step, assume

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots \longrightarrow I^n$$

is exact. Take $Y := \operatorname{coker}(I^{n-1} \to I^n)$. Since $\mathcal{A}$ has enough injectives, there exists an I^{n+1} injective and a monomorphism $Y \hookrightarrow I^{n+1}$. Then the sequence

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X & \longrightarrow & I^0 & \longrightarrow & \dots & \longrightarrow & I^n & \longrightarrow & I^{n+1} \\ & & & & & & & & & \searrow\!\!\!\!\searrow \quad Y \quad \nearrow & \end{array}$$

is exact since

$$\operatorname{im}(I^{n-1} \to I^n) = \ker(I^n \to \operatorname{coker}(I^{n-1} \to I^n)) = \ker(I^n \to Y) = \ker(I^n \to I^{n+1})$$

where the third equality follows since postcomposition with a monomorphism does not affect the kernel. This completes the induction. □

Lemma 11.2.5 · *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let*

$$\mathcal{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{B} \quad (F \dashv U)$$

be an adjunction where F is exact. Then U preserves injective objects.

Proof. Let I be injective object of $\mathscr{B}$ and let $A \rightarrowtail B$ be a monomorphism in $\mathscr{A}$. Then $FA \rightarrowtail FB$ is a monomorphism by exactness of F, so the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{A}}(B,UI) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(A,UI) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ \mathrm{Hom}_{\mathscr{B}}(FB,I) & \twoheadrightarrow & \mathrm{Hom}_{\mathscr{B}}(FA,I) \end{array}$$

commutes, whence the top map is surjective. □

Definition 11.2.6 · An abelian category $\mathscr{A}$ has *functorial injectives* if there is a functor $F : \mathscr{A} \to \mathbf{Fun}([1], \mathscr{A})$ such that:

wording suggests this is property but it is structure!

(i) $F_0 = \mathrm{id}_{\mathscr{A}}$,

(ii) each $F_1(X)$ is injective, and

(iii) $F_0(X) \hookrightarrow F_1(X)$ for all $X \in \mathscr{A}$.

We write $X \to I(X)$ for this functor. Dually, one defines the notion of *functorial projectives*.

Remark 11.2.7 · If $\mathscr{A}$ has functorial injectives, then it has functorial injective resolutions.

interpretation?

Theorem 11.2.8 · *(i) The categories ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ of left and right R-modules have functorial injectives and projectives.*

(ii) The category $\mathbf{PAb}(\mathscr{C})$ of abelian presheaves on a category $\mathscr{C}$ has functorial injectives and projectives.

(iii) The category $\mathbf{Ab}(X)$ of abelian sheaves on a space X has functorial injectives.

Proof (sketch). (i) Free modules give functorial projectives: $F(M) := R^{\oplus M} \twoheadrightarrow M$. For the injective case, consider the map

$$(-)^\vee : {}_R\mathbf{Mod}^{\mathrm{op}} \to \mathbf{Mod}_R, \quad M \mapsto \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

where $M^\vee$ is a right module via the action,

$$M^\vee \times R \to M^\vee, \quad (\varphi, e) \mapsto (m \mapsto \varphi(r \cdot m)).$$

Note that the functor $(-)^\vee$ is exact since $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group. Moreover

$$\mathrm{ev} : M \to (M^\vee)^\vee, \quad m \mapsto (\varphi \mapsto \varphi(m))$$

is injective. For suppose $m \neq 0$, if we take $\psi : \mathbb{Z}m \to \mathbb{Q}/\mathbb{Z}$ with $\psi(m) \neq 0$ and extend this to $\varphi : M \mapsto \mathbb{Q}/\mathbb{Z}$, then $\mathrm{ev}(m) \neq 0$ since $\varphi(m) \neq 0$. Now $F(M^\vee) \twoheadrightarrow M^\vee$ (recall F is a functorial projective) gives

$$M \overset{\mathrm{ev}}{\rightarrowtail} (M^\vee)^\vee \rightarrowtail F(M^\vee)^\vee$$

and we claim that $F(M^\vee)^\vee$ is injective. Indeed, if S is a set, then $F(S)^\vee$ is injective since

$$\begin{aligned} {}_R\mathrm{Hom}(N, F(S)^\vee) &= {}_R\mathrm{Hom}(N, \mathrm{Hom}_{\mathbb{Z}}(F(S), \mathbb{Q}/\mathbb{Z})) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(F(S) \otimes_R N, \mathbb{Q}/\mathbb{Z}) && \text{(Exercise 11.2)} \\ &= \mathrm{Hom}_{\mathbb{Z}}(N^{\oplus S}, \mathbb{Q}/\mathbb{Z}) \\ &= (N^\vee)^S, \end{aligned}$$

which we saw was exact.

(ii) Let F be an abelian presheaf on a category $\mathscr{C}$. For projectives, use $\bigoplus_{U\in \operatorname{ob}\mathscr{C}} \underline{\mathbb{Z}}_U^{\oplus F(U)} \twoheadrightarrow F$ (supposedly easy to prove; this not longer works in $\mathbf{Ab}(X)$ since $\underline{\mathbb{Z}}_U$ is no longer projective as we will see in Homework 6). For injectives, this is certainly true for $\mathbf{PAb}(\mathscr{C}^{\mathrm{disc}}) \cong \prod_{X\in \operatorname{ob}\mathscr{C}} \mathbf{Ab}$ by part (i). The inclusion of categories $i : \mathscr{C}^{\mathrm{disc}} \hookrightarrow \mathscr{C}$ gives rise to

$$i^* : \mathbf{PAb}(\mathscr{C}) \to \mathbf{PAb}(\mathscr{C}^{\mathrm{disc}}), \quad F \mapsto F \circ i$$

with right adjoint $i_* = \operatorname{Ran}_i$. Then i^* is exact, so i_* preserves injectives. Thus the functorial injective $i^*F \hookrightarrow I(i^*F)$ gives

reference pullback/pushforward adjunction

$$F \longrightarrow i_*i^*F \longrightarrow i_*I(i^*F).$$

where the first map is the unit of the adjunction $i^* \dashv i_*$. The unit is monic since i^* is faithful and the second map is since i_* is left exact.

(iii) This again holds in $\mathbf{Ab}(\mathscr{C}^{\mathrm{disc}}) \cong \prod_{x\in X} \mathbf{Ab}$ by (i). The inclusion $i : X^{\mathrm{disc}} \hookrightarrow X$ gives adjunction

$$\mathbf{Ab}(X) \underset{i_*}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Ab}(X^{\mathrm{disc}})$$

with i^* exact and faithful (look at stalks). Now run the same argument as in part (ii). □

Example 11.2.9 · If $R = \mathbb{Z}$ then $F(S)^\vee = (\mathbb{Z}^{\oplus S})^\vee = (\mathbb{Z}^\vee)^S = (\mathbb{Q}/\mathbb{Z})^S$.

Exercises

if we decide to do this, cite properly in intro w/ permission Remy

Exercise 11.1 (Products of injectives are injective) · Let $\mathscr{A}$ be an abelian category. Show that the subcategory of injective objects (resp. projective objects) is closed under products (resp. direct sums) that exist in $\mathscr{A}$.

Exercise 11.2 (Hom is right adjoint to the tensor product) · For rings A and B, a (A, B)-*bimodule* is an abelian group M which is a left A-module and a right B-module such that $a(mb) = (am)b$ for all $a \in A$, $m \in M$ and $b \in B$. Let M be an (A, B)-bimodule, let N be a (B, C)-bimodule and let P be an (A, C)-bimodule. Construct isomorphisms

$$\begin{aligned} {}_A\operatorname{Hom}(M \otimes_B N, P) &\cong {}_B\operatorname{Hom}(N, {}_A\operatorname{Hom}(M, K)) \\ \operatorname{Hom}_C(M \otimes_B N, P) &\cong \operatorname{Hom}_B(M, \operatorname{Hom}_C(N, K)) \end{aligned}$$

of (C, C)-bimodules and (A, A)-bimodules respectively. Restricting to elements on which the two C-actions (resp. A-actions) agree, deduce isomorphisms of abelian groups

$${}_B\operatorname{Hom}_C(N, {}_A\operatorname{Hom}(M, K)) \cong {}_C\operatorname{Hom}(M \otimes_B N, P) \cong {}_A\operatorname{Hom}_B(M, \operatorname{Hom}_C(N, K)).$$

LECTURE 12

Derived functors, sheaf cohomology

This week's lecture was Dr. Soumya Sankar.

12.1 Derived functors

Let $C^\bullet$ be a cochain complex. The kernels $Z^i := \ker(d^i : C^i \to C^{i+1})$ are called the *cocycles* of the complex, and the images $B^i := \operatorname{im}(d^{i-1} : C^{i-1} \to C^i)$ are its *coboundaries*. Check that the universal property of the kernel gives a natural map $B^i \to Z^i$.

Definition 12.1.1 · The cokernel $\operatorname{coker}(B^i \to Z^i)$ is the *ith cohomology* of $C^\bullet$. It is denoted by $H^i(C^\bullet)$. One checks that in abelian groups $H^i(C^\bullet)$ is the quotient Z^i/B^i.

this is not so clear...

Given a morphism $C^\bullet \to D^\bullet$, taking the ith cohomology functorially induces a map $H^i(C^\bullet) \to H^i(D^\bullet)$.

If $C^\bullet$ is exact, $H^i(C^\bullet) = 0$ for all i.

Exercise 12.1.2 · Let $C^\bullet = 0 \to X \to I^0 \to I^1 \to \ldots$ be an injective resolution. Then $H^0(C^\bullet) = X$ and $H^i(C^\bullet) = 0$ for $i > 0$.

Proposition 12.1.3 · *Let $C^\bullet \in \mathbf{Ch}(\mathscr{A})$ be a cochain complex for $\mathscr{A}$ an abelian category. For all i there is a natural map*

$$\varphi : \operatorname{coker} d^{i-1} \to \ker d^{i+1}.$$

Moreover, the map recovers cohomology in degrees i and $i+1$: there are isomorphisms $H^i(C^\bullet) \cong \ker\varphi$ and $H^{i+1}(C^\bullet) \cong \operatorname{coker}\varphi$.

Proof. Notice that d^i factors via $\ker d^{i+1}$ and via $\operatorname{coker} d^{i-1}$ via the universal properties of the kernel and cokernel respectively. Since $C^i \to \operatorname{coker} d^{i-1}$ is an epimorphism and $d^{i+1} \circ d^i = 0$, the composition $\operatorname{coker} d^{i-1} \to C^{i+1} \to C^{i+2}$ is also zero, so the map $\operatorname{coker} d^{i-1} \to C^{i+1}$ factors via $\ker d^{i+1}$. It may be helpful to stare at the following diagram:

$$\begin{array}{ccccc} C^i & \xrightarrow{d_C^i} & C^{i+1} & \xrightarrow{d_C^{i+1}} & C^{i+2} \\ \downarrow & \nearrow & \uparrow & & \\ \operatorname{coker} d_C^{i-1} & \dashrightarrow & \ker d^{i+1}. & & \end{array}$$

We now obtain a unique map $\ker d^i \to \ker\varphi$: the map $\ker d^i \to C^i \to \operatorname{coker} d^{i-1} \to \ker d^{i+1} \to C^{i+1}$ is zero, and thus the map $\ker d^i \to C^i \to \operatorname{coker} d^{i-1} \to \ker d^{i+1}$ is also zero ($\ker d^{i+1} \to C^{i+1}$ is a monomorphism).

$H^i(C^\bullet) \to \operatorname{coker} d^{i-1}$ □

Lemma 12.1.4 (Long exact sequence of cohomology) · *Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence of chain complexes. Then there is a canonical long exact sequence*

$$\cdots \to H^{i-1}(A^\bullet) \to H^{i-1}(B^\bullet) \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to \cdots$$

Proof. From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{i+1} & \longrightarrow & B^{i+1} & \longrightarrow & C^{i+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^{i+2} & \longrightarrow & B^{i+2} & \longrightarrow & C^{i+2} & \longrightarrow & 0 \end{array}$$

and the snake lemma we get an exact sequence $0 \to \ker d_A^{i+1} \to \ker d_B^{i+1} \to \ker d_C^{i+1}$. By shifting indices and again using the snake lemma we obtain the exact sequence

$$\operatorname{coker}(d_A^{i-1}) \to \operatorname{coker}(d_B^{i-1}) \to \operatorname{coker}(d_C^{i-1}) \to 0.$$

We can now apply the snake lemma to the diagram

$$\begin{array}{ccccccccc} & & \operatorname{coker} d_A^{i-1} & \longrightarrow & \operatorname{coker} d_B^{i-1} & \longrightarrow & \operatorname{coker} d_C^{i-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker d_A^{i+1} & \longrightarrow & \ker d_B^{i+1} & \longrightarrow & \ker d_C^{i+1}. & & \end{array}$$

By Proposition 12.1.3 we thus obtain the long exact sequence of cohomology. □

Definition 12.1.5 · Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor between abelian categories and suppose $\mathscr{A}$ has enough injectives. Define the *ith right derived functor* $\mathrm{R}^i F : \mathscr{A} \to \mathscr{B}$ as follows: for an object X of $\mathscr{A}$, choose an injective resolution

$$0 \to X \to I^0 \to I^1 \to I^2 \to \dots,$$

apply F levelwise to the resolution to obtain the complex

$$0 \to FI^0 \to FI^1 \to FI^2 \to \dots$$

and set $\mathrm{R}^i F(X) := H^i(F(I^\bullet))$ to be the ith cohomology of this complex.

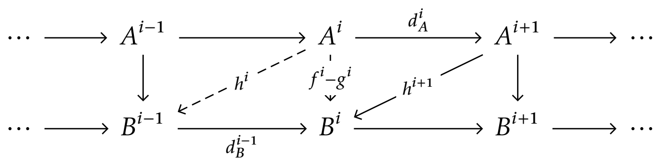
In the remainder of this lecture, we will verify that this definition makes sense and we apply it to introduce *sheaf cohomology*. Additional exercise 12.3 shows the functoriality of this construction.

12.2 Homotopies and quasi-isomorphisms

Definition 12.2.1 · Let $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ be morphisms of cochain complexes. A *homotopy* between $f^\bullet$ and $g^\bullet$ is a collection of maps $h^i : A^i \to B^{i-1}$ such that

$$f^i - g^i = d_B^{i-1} h^i + h^{i+1} d_A^i.$$

The maps $f^\bullet$ and $g^\bullet$ are *homotopic*, we write $f^\bullet \sim g^\bullet$. It helps to keep the following 'parallelogram diagram' in mind:



If a morphism is homotopic to the zero morphism, it is called *nullhomotopic*.

Definition 12.2.2 · A map $f^\bullet : A^\bullet \to B^\bullet$ is a *homotopy equivalence* if there exists a map $g^\bullet : B^\bullet \to A^\bullet$ such that $g^\bullet \circ f^\bullet \sim \mathrm{id}_{g^\bullet}$ and $g^\bullet \sim \mathrm{id}_{f^\bullet}$.

Definition 12.2.3 · A map $f^\bullet : A^\bullet \to B^\bullet$ is a *quasi-isomorphism* if the induced map

$$H^i f^\bullet : H^i A^\bullet \to H^i B^\bullet$$

is an isomorphism for all $i \in \mathbb{Z}$.

Exercise 12.2.4 · Let $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ be morphisms of cochain complexes.

(a) If $f^\bullet \sim g^\bullet$ then $f^\bullet$ and $g^\bullet$ induce the same maps on homology.

(b) If $f^\bullet$ is a homotopy equivalence then it is a quasi isomorphism.

(c) If $F : \mathscr{A} \to \mathscr{B}$ is an additive functor between abelian categories and h is a homotopy between $f^\bullet$ and $g^\bullet$ then $F(h)$ is a homotopy between $F(f^\bullet)$ and $F(g^\bullet)$[1].

(d) If f is a homotopy equivalence and F is additive then $F(f)$ is a quasi-isomorphism.

Lemma 12.2.5 · *Let $C^\bullet$ be an exact cocomplex and*

$$I^\bullet : 0 \to I^k \to I^{k+1} \to I^{k+2} \to \cdots$$

be a cocomplex of injective objects (we call a cocomplex which is zero for all i smaller or larger than some $k \in \mathbb{Z}$ a bounded *cocomplex). Then any map of complexes $f^\bullet : C^\bullet \to I^\bullet$ is nullhomotopic.*

Proof. We want to construct maps $h^i : C^i \to I^{i-1}$ that satisfy the homotopy condition with respect to $f^\bullet$ and 0:

$$f^i - 0 = d_B^{i-1} h^i + h^{i+1} d_I^i$$

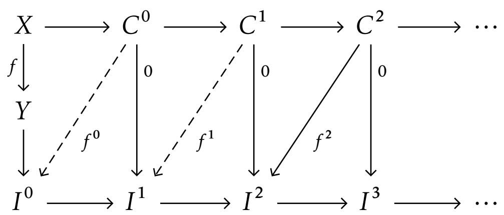
for all $i \in \mathbb{Z}$. We proceed inductively. For $i < k$ we let $h^i = 0$. Assume now that h^j is defined for all $j \leqslant i$ for some $i \in \mathbb{Z}$. Some computation yields an insight. We have

$$\begin{aligned}
(f^i - d_I^{i-1} h^i) \circ d_C^{i-1} &= f^i \circ d_C^{i-1} - d_I^{i-1} \circ h^i \circ d_C^{i-1} \\
d_I^{i-1} f^{i-1} - d_I^{i-1} \circ h^i \circ d_C^{i-1} &= d_I^{i-1}(f^{i-1} - h^i \circ d_C^{i-1}) \\
d_I^{i-1} \circ d_I^{i-2} \circ h^{i-1} &= 0
\end{aligned}$$

the last step by the inductive hypothesis. We see that $f^i - d_I^{i-1}$ factors via the cokernel of d_C^{i-1} which by exactness is the image of .... . □ finish

Corollary 12.2.6 · *Let $0 \to Y \to I^0 \to I^1 \to \cdots$ be a bounded cocomplex of injectives, and let $0 \to X \to C^0 \to C^1 \to ...$ be an exact cocomplex. Then any map $f : X \to Y$ extends to the cocomplexes uniquely up to homotopy.*

Proof. Construct the following solid commutative diagram



giving a map of cocomplexes from $X \to C^0 \to C^1 \to C^2 \to \cdots$ to $I^0 \to I^1 \to I^2 \to \cdots$. By Lemma 12.2.5 □ rest.

[1] This is a slight abuse of notation. A homotopy is not a morphism of complexes, but a set of maps h^i for $i \in \mathbb{Z}$. Here $F(h)$ means we apply F to each h^i

Corollary 12.2.7 · *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories, and suppose $\mathcal{A}$ has enough injectives. Then the derived functors $\mathrm{R}^i F$ are well defined for all i.*

Proof. Suppose $0 \to X \to I^\bullet$ and $0 \to X \to J^\bullet$ are injective resolutions. Then the identity map $\mathrm{id}_X : X \to X$ extends to two maps $f^\bullet : I^\bullet \to J^\bullet$ and $g^\bullet : J^\bullet \to I^\bullet$. Since either composition extends the identity on X uniquely up to homotopy, and the identity on the complexes extends the identity on X, the map $f^\bullet$ is a homotopy equivalence, so $F(f)$ is a quasi-isomorphism by Exercise 12.2.4, and thus $H^i(F(I^\bullet)) \cong H^i(F(J^\bullet))$. . □

prove functoriality of the derived functors

12.3 Sheaf cohomology

The *sections functor* $\Gamma(U, -) : \mathbf{Ab}(X) \to \mathbf{Ab}$, $\mathcal{F} \mapsto \mathcal{F}(U)$ is left exact for any open U of a space X. Hence the right derived functors $\mathrm{R}^i\Gamma(U, -)$ exist and are well-defined.

Definition 12.3.1 · Let X be a topological space and let $U \subseteq X$ be an open subset. The ith *sheaf cohomology* is the ith right derived functor $\mathrm{R}^i\Gamma(U, -)$. It is denoted $H^i(U, -)$.

of U?

Example 12.3.2 · In Homework 6, Exercise 2, you constructed a short exact sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0$$

of sheaves on $\mathbb{C}^n$. For some fixed open $U \subseteq \mathbb{C}^n$, the sheaf cohomology $H^i(U, -)$ is the ith right derived functor of $\Gamma(U, -)$. The short exact sequence gives rise to a long exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(U, \underline{\mathbb{Z}}) \longrightarrow H^0(U, \mathcal{O}) \longrightarrow H^0(U, \mathcal{O}^\times) \\ \hookrightarrow H^1(U, \underline{\mathbb{Z}}) \longrightarrow H^1(U, \mathcal{O}) \longrightarrow H^1(U, \mathcal{O}^\times) \longrightarrow \dots \end{array}$$

LECTURE 13

Acyclic resolutions, supports

13.1 Acyclic resolutions

'"For the benefit of those millenials who believe that the Godement resolution is one of the founding documents of the United Nations, here is a translation of the above construction into contemporary language." [HV19, Remark 7.1]'

In this section, let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and assume $\mathcal{A}$ has enough injectives.

Definition 13.1.1 · Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Then an object A of $\mathcal{A}$ is *F-acyclic* if $\mathrm{R}^i F(A) = 0$ for $i > 0$.

Example 13.1.2 · If I is an injective object of $\mathcal{A}$, then it is F-acyclic for any F. Indeed, take the injective resolution

$$0 \to I \to \underline{I \to 0 \to 0 \to \dots}$$

of I. Then

$$\mathrm{R}^i F(I) = H^i(F(I) \to 0 \to 0 \to \dots) = \begin{cases} F(I) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

The following lemma says that the right-derived functors of a left exact functor F can also be computed using F-acyclic resolutions rather than injective resolutions.

Lemma 13.1.3 · *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor, let A be an object of $\mathcal{A}$ and let*

$$0 \to A \to C^0 \to C^1 \to \dots$$

be a resolution of A with F-acyclic C^i for all i. Then $\mathrm{R}^i F(A) \cong H^i(F(C^\bullet))$ for all i.

Proof. We proceed by induction on i.

For $i = 0$, left exactness gives

$$H^0(F(C^\bullet)) \cong \ker(F(C^0) \to F(C^1)) \cong F(A) \cong \mathrm{R}^0 F(A).$$

(We have not shown this yet, but it is not too hard to show that $\mathrm{R}^0 F \cong F$.)

Let $B := \operatorname{im}(C^0 \to C^1) \cong \ker(C^1 \to C^2)$. By breaking of the resolution using B, we get exact sequences

$$0 \to A \to C^0 \to B \to 0 \tag{13.1}$$

and

$$0 \to B \to C^1 \to C^2 \to \dots \tag{13.2}$$

The short exact sequence (13.1) gives a long exact sequence

ref long exact sequence

$$0 \to F(A) \to F(C^0) \to F(B) \to \mathrm{R}^1F(A) \to \underbrace{\mathrm{R}^1F(C^0)}_{=0} \to \mathrm{R}^1F(B) \to \mathrm{R}^2F(A) \to \underbrace{\mathrm{R}^2F(C^0)}_{=0} \to \dots$$

in which every third term is (eventually) zero since C^0 is F-acyclic; hence we obtain isomorphisms $\mathrm{R}^iF(B) \cong \mathrm{R}^{i+1}F(A)$ by exactness. The long exact sequence (13.2) gives an exact sequence

$$0 \to F(B) \to F(C^1) \to F(C^2)$$

by left exactness of F. Thus, we have

$$\mathrm{R}^1F(A) \cong \frac{F(B)}{F(C^0)} \cong \frac{\ker(F(C^1) \to F(C^2))}{\mathrm{im}(F(C^0) \to F(C^1))} \cong H^1(F(C^\bullet)),$$

proving the result for $i = 1$.

For $i > 1$, assume the result for $i-1$ (and for all A and $C^\bullet$). Applying the induction hypothesis to the F-acyclic resolution (13.2) of B, we find

$$\mathrm{R}^iF(A) \cong \mathrm{R}^{i-1}F(B) \cong H^{i-1}(F(C^1) \to F(C^2) \to \dots) = H^i(F(C^0) \to F(C^1) \to \dots). \qquad \square$$

So far, this lemma does not say anything specific about sheaves. The next lemma will be an important example for us. Recall that a sheaf $\mathcal{F}$ is *flasque* if the restriction $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective for opens $U \subseteq V$. We will use the result of the following exercise, which appears as Additional exercise 12.4.

Exercise 13.1.4 · Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of abelian sheaves on a space X. Show the following:

(i) If $\mathcal{F}$ is flasque and $U \subseteq X$ is open, then $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$ is exact. (In other words, the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is exact.)

(ii) If $\mathcal{F}$ and $\mathcal{G}$ are flasque, then so is $\mathcal{H}$.

Lemma 13.1.5 · *Let $\mathcal{F}$ be a flasque abelian sheaf on a space X. Then $\mathcal{F}$ is $H^0(U,-)$-acyclic for any open $U \subseteq X$, that is, $H^i(U,\mathcal{F}) = 0$ for $i > 1$.*

Proof. Choose an injection $\mathcal{F} \hookrightarrow \mathcal{J}$ into an injective sheaf and let $\mathcal{H}$ be the quotient, giving a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{J} \to \mathcal{H} \to 0.$$

Injective sheaves are flasque (Example 11.1.10), so by Exercise 13.1.4, $\mathcal{H}$ is too and

$$0 \to \mathcal{F}(U) \to \mathcal{J}(U) \to \mathcal{H}(U) \to 0$$

is exact for any open $U \subseteq X$. Since injective sheaves are acyclic so $H^i(U,\mathcal{J}) = 0$, we get $H^i(U,\mathcal{F}) = 0$ for $i > 0$ and $H^i(U,\mathcal{F}) \cong H^{i-1}(U,\mathcal{H})$ for $i > 1$ from the long exact sequence. Since $\mathcal{H}$ is flasque, we conclude by induction on i. (This argument is similar to what we did in the proof of Lemma 13.1.3.) $\square$

Example 13.1.6 · If X is discrete, then any sheaf on X is flasque, so $H^i(X,\mathcal{F}) = 0$ for all $i > 0$ and all abelian sheaves $\mathcal{F}$ on X.

Example 13.1.7 · If $f : Y \to X$ is continuous and $\mathcal{F}$ is a flasque abelian sheaf on Y, then $f_*\mathcal{F}$ is flasque: if $U \subseteq V \subseteq X$ are open, then the diagram

$$\begin{array}{ccc} f_*\mathcal{F}(V) & \longrightarrow & f_*\mathcal{F}(U) \\ \| & & \| \\ \mathcal{F}(f^{-1}(V)) & \twoheadrightarrow & \mathcal{F}(f^{-1}(U)) \end{array}$$

commutes, so surjectivity of the bottom map (by flasqueness of $\mathcal{F}$) implies surjectivity of the top map.

Remark 13.1.8 · Recall that we constructed enough injectives in $\mathbf{Ab}(X)$ as follows: take $f : X^{\text{disc}} \hookrightarrow X$ and choose an injection $f^*\mathcal{F} \hookrightarrow \mathcal{J}$ into an injective in $\mathbf{Ab}(X^{\text{disc}})$. Then the composite

$$\mathcal{F} \hookrightarrow f_* f^* \mathcal{F} \hookrightarrow f_* \mathcal{J}$$

gives our required injection into an injective object. But the map $\mathcal{F} \hookrightarrow f_* f^* \mathcal{F}$ is already an injection into a flasque sheaf. Iterating this construction gives a canonical flasque resolution

$$0 \to \mathcal{F} \to f_* f^* \mathcal{F} \to f_* f^* (\operatorname{coker}(\mathcal{F} \to f_* f^* \mathcal{F})) \to \dots,$$

which is called the *Godement resolution* of $\mathcal{F}$. This resolution has some advantages: it is very explicit, functorial and additive.

13.2 Supports

Definition 13.2.1 · Let $\mathcal{F}$ be an abelian sheaf on X and let $s \in \mathcal{F}(U)$ be a section. Then the support of s is the set $\operatorname{supp}(s) := \{ x \in U \mid s_x \neq 0 \}$.

Lemma 13.2.2 · *If $s \in \mathcal{F}(U)$ is a section, then $\operatorname{supp}(s) \subseteq U$ is closed.*

Proof. The locus where the two sections $0, s : U \to \operatorname{sp}(\mathcal{F})$ agree is open since the diagonal

$$\Delta_{\operatorname{sp}(F)/X} \subseteq \operatorname{sp}(F) \times_X \operatorname{sp}(\mathcal{F})$$

is open. □ why?

Definition 13.2.3 · For a topological space X, define a functor

$$H^0_c(X, -) : \mathbf{Ab}(X) \to \mathbf{Ab}, \quad \mathcal{F} \mapsto \{ s \in \mathcal{F}(X) \mid \operatorname{supp}(s) \text{ is compact} \},$$

sending a sheaf to its global sections with compact support. This functor is left exact and the derived functors $H^i_c(X, -) := \mathrm{R}^i H^0_c(X, -)$ are the *sheaf cohomology with compact support*.

The following lemma shows that ordinary sheaf cohomology and sheaf cohomology with compact support agree for all sheaves on compact spaces.

Lemma 13.2.4 · *Let $\mathcal{F}$ be an abelian sheaf on a compact space X. Then $H^i(X, \mathcal{F}) \cong H^i_c(X, \mathcal{F})$ for all $i \geqslant 0$.*

Proof. By Lemma 13.2.2, the support of any global section $s \in \mathcal{F}(X)$ is closed in the compact space X, so also compact. Hence we have $H^0_c(X, -) = \Gamma(X, -) = H^0(X, -)$, and then also their derived functors $H^i_c(X, -)$ and $H^i(X, -)$ agree for all $i \geqslant 0$. □

We will derive properties of cohomology H^i and compactly supported cohomology H^i_c in parallel. A more general unified strategy using 'families of supports' is presented in [Bre97, § II.9].

For a sheaf $\mathcal{F}$ on a space X and any subspace $i : Y \hookrightarrow X$, we write $\mathcal{F}|_Y$ for the *restriction* of $\mathcal{F}$ to Y, which is defined to be the pullback $i^*\mathcal{F}$ of $\mathcal{F}$ along i.

Definition 13.2.5 · An abelian sheaf $\mathcal{F}$ on a space X is *soft* (resp. *c-soft*) if the restriction $\mathcal{F}(X) \to \mathcal{F}|_Z(Z)$ is surjective for every closed (resp. compact closed) subset $Z \subseteq X$.

By abuse of notation, we write $\mathcal{F}(W) = \mathcal{F}|_W(W)$ when $W \subseteq X$ is locally closed.

Remark 13.2.6 · Some authors (such as [Ive86]) say 'soft' for what we call 'c-soft'.

We will show that soft (resp. c-soft) sheaves are $H^0(X, -)$-acyclic (resp. $H^0_c(X, -)$-acyclic) under suitable assumptions on the space X.

13.3 Paracompactness

> 'We're diving into point-set topology, it's happening: it's no longer a category theory course.'

Recall that a space X is *paracompact* if every open cover $X = \bigcup_{i \in I} U_i$ has a refinement $\bigcup_{j \in J} V_j = X$ (that is, there exists a function $\varphi : J \to I$ with $V_j \subseteq U_{\varphi(j)}$ for all $j \in J$) that is locally finite: every point $x \in X$ has an open neighbourhood U meeting finitely many V_j.

Example 13.3.1 · All compact spaces, metrisable spaces and CW-complexes are paracompact. Most authors *assume* manifolds to be paracompact.

We will use the following point-set topological result about paracompact spaces.

Lemma 13.3.2 ([Munoo, Lemma 41.6]) · *If X is a paracompact Hausdorff space and $\bigcup_{i \in I} U_i = X$ is an open cover, then there exists a locally finite cover $\bigcup_{i \in I} V_i = X$ with $\overline{V_i} \subseteq U_i$ for all $i \in I$.*

Recall also that a space X is *locally compact* if every point has a *compact neighborhood*, i.e., if there exists for every $x \in X$ an open U containing x and a compact set $K \supset U$.

Proposition 13.3.3 · *Let X be a Hausdorff space, let $i : Z \hookrightarrow X$ be a closed subspace and let $\mathcal{F}$ be an abelian sheaf on X. Then the map*

$$\operatorname*{colim}_{Z \subseteq U,\, U \text{ open}} \mathcal{F}(U) \xrightarrow{(-)|_Z} \mathcal{F}(Z) \tag{13.3}$$

is an isomorphism if:

(i) X is paracompact, or

(ii) X is locally compact and Z is compact.

Proof (sketch). Recall that $\operatorname{colim}_{Z \subseteq U} \mathcal{F}(U) = (i^{\circledast}\mathcal{F})(Z)$ (see Proposition 5.2.1) and we defined $\mathcal{F}(Z) := (i^*\mathcal{F})(Z)$.

The map (13.3) is always injective: suppose $s \in \mathcal{F}(U)$ is a section with $s|_Z = 0$. Then the support of s is a closed subset (by Lemma 13.2.2) of the complement $U \smallsetminus Z$, so $V := U \smallsetminus \operatorname{supp}(s)$ is an open neighbourhood of Z such that $s|_V = 0$.

For surjectivity, we treat the cases separately:

(i) If $s \in \mathcal{F}(Z)$ is a section, then it *locally* extends to an open neighbourhood: there exist opens $U_i \subseteq X$ such that $Z \subseteq \bigcup_{i \in I} U_i$ and sections $t_i \in \mathcal{F}(U_i)$ with $t|_{Z \cap U_i} = s|_{Z \cap U_i}$ for all $i \in I$. Without loss of generality, we may assume that the open cover $(U_i)_{i \in I}$ is locally finite since closed subspaces of paracompact spaces are paracompact. (This is [Munoo, Theorem 41.2]; the idea of the proof is to add $X \smallsetminus Z$ to the cover and restrict the refined cover by intersecting with Z.) Using Lemma 13.3.2, choose an open cover $Z = \bigcup_{i \in I} V_i = X$ with $\overline{V_i} \subseteq U_i$ for all $i \in I$. Set

$$W := \{ x \in X \mid x \in \overline{V_i} \cap \overline{V_j} \implies t_{i,x} = t_{j,x} \},$$

which contains Z and write $J(x) := \{ i \in I \mid x \in \overline{V_i} \}$ for a point $x \in X$. Then for all x, the set $J(x)$ is finite and there is an open neighbourhood U of x such that $J(y) \subseteq J(x)$ for all $y \in U$: indeed, there exists an open neighbourhood V of x meeting finitely only many U_i, and we can take

$$U := V \smallsetminus \bigcup_{\substack{i \in I \\ U_i \cap V \neq \varnothing \\ x \notin V_i}} \overline{V_i}.$$

If $x \in W$, the sections $t_{i,x}$ for $i \in J(x)$ all agree, so the same holds in a neighbourhood of x since $J(x)$ is finite. Hence W is open and the t_i glue to a well-defined section $t \in \mathcal{F}(W)$.

(ii) Since X is locally compact, there exists a compact neighbourhood W of Z in X. Replace X by W and use part (i). □

Corollary 13.3.4 · *If $\mathscr{F}$ is a flasque abelian sheaf on X, then $\mathscr{F}$ is:*

(i) soft if X is paracompact Hausdorff, and

(ii) c-soft if X is locally compact Hausdorff.

Proposition 13.3.5 · *Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be a short exact sequence of abelian sheaves on a space X.*

(i) If X is paracompact Hausdorff and $\mathscr{F}$ is soft, then the short sequence

$$0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X) \to 0$$

is exact.

(ii) If X is locally compact Hausdorff and $\mathscr{F}$ is c-soft, then the short sequence

$$0 \to H^0_c(X,\mathscr{F}) \to H^0_c(X,\mathscr{G}) \to H^0_c(X,\mathscr{H}) \to 0$$

is exact.

Before proving the proposition, we note the following consequence.

Corollary 13.3.6 · *A soft (resp. c-soft) sheaf on a paracompact (resp. locally compact) Hausdorff space X is $H^0(X,-)$-acyclic (resp. $H^0_c(X,-)$-acyclic).*

Proof. Similar to the proof of Lemma 13.1.5, using a result similar to Exercise 13.1.4 for softness or c-softness instead of flasqueness. □

Proof (of Proposition 13.3.5, sketch). (i) Exactness on the left is general, so it remains to prove that $\mathscr{G}(X) \to \mathscr{H}(X)$ is surjective. Let $s \in \mathscr{H}(Z)$ be a section and let $\bigcup_{i\in I} U_i = X$ be a locally finite cover with $s_i \in \mathscr{G}(U_i)$ lifting $s|_{U_i}$ for all $i \in I$. By Lemma 13.3.2, choose a locally finite cover $\bigcup_{i\in I} V_i = X$ with $\overline{V_i} \subseteq U_i$ for all $i \in I$. *Now we are going to do one of the most ugly tricks in the book*: Put all well-order on I and let $I^{\triangleright} := I \cup \{\infty\}$ be the successor of I, that is, with $i < \infty$ for all $i \in I$. For $i \in I^{\triangleright}$ set $W_i := \bigcup_{j<i} \overline{V}_j$; in particular, $W_\infty = X$. Since the open cover $(U_i)_{i\in I}$ is locally finite, each W_i is closed. Using transfinite induction we define $t_i \in \mathscr{G}(W_i)$ for all $i \in I^{\triangleright}$ lifting $s|_{W_i}$ such that $t_i|_{W_j} = t_j$ if $j \leqslant i$.

For the smallest element i_0 of $I^{\triangleright}$, we have $W_{i_0} = \varnothing$, so we must choose $t_{i_0} = 0 \in \mathscr{F}(W_{i_0}) = \mathscr{F}(\varnothing) = 0$, and this clearly lifts $s|_{W_i} = 0$.

In the successor case $i = j+1$ for $j \in I$, then t_j and s_i both lift $s|_{W_j \cap \overline{V}_j}$, so their difference is in $\mathscr{F}(W_j \cap \overline{V}_j)$ (by exactness of pullback, Lemma 9.1.1, restriction to a closed subspace is exact). This difference extends to some $t_{ij} \in \mathscr{F}(X)$ since $\mathscr{F}$ is soft, and then $t_j \in \mathscr{G}(W_j)$ and $s_i - t_{ij}|_{\overline{V}_j} \in \mathscr{G}(\overline{V}_j)$ glue to some $t_i \in \mathscr{G}(W_i)$. (One of the additional exercises shows you can do this.) which one?

Finally, if i is a limit ordinal, then glue t_j for $j < i$ to get t_i.

(ii) If $s \in H^0_c(X,\mathscr{H})$ is a section with compact support, then apply part (i) to a compact neighbourhood W of $\operatorname{supp}(s)$ to get $t \in \mathscr{G}(W)$ lifting $s|_W$. On the boundary ∂W of W, we have $s|_{\partial W} = 0$, so $t|_{\partial W}$ is in $\mathscr{F}(\partial W)$. Lift to a global section of $\mathscr{F}$, subtract and extend by zero. □

The conclusion of all the work we did is the following picture for abelian sheaves on a Hausdorff space X:

$$
\begin{array}{ccccc}
 & & & \text{soft} \Longrightarrow H^0(X,-)\text{-acyclic} \\
 & & \nearrow & \\
\text{injective} & \Rightarrow \text{flasque} & & \\
 & & \searrow & \\
 & & & \text{c-soft} \Rightarrow H^0_c(X,-)\text{-acyclic}
\end{array}
$$

The top implications hold under the assumption that X is paracompact and the bottom implications under the assumption that X is locally compact.

LECTURE 14

Soft sheaves, exponential sequence, higher pushforwards

14.1 Soft sheaves of rings

We will show that the sheaves $C^0(-,\mathbb{R}) = h_{\mathbb{R}}$ and $C^\infty(-,\mathbb{R})$ (on a manifold) are soft.

Proposition 14.1.1 · *Let X be a Hausdorff space.*

(i) If X is paracompact then $C^0(-,\mathbb{R}) = h_{\mathbb{R}}$ is soft.

(ii) If X is locally compact, then $C^0(-,\mathbb{R})$ is c-soft.

Proof. (i) Let Z be closed and $s \in C^0(-,\mathbb{Z})|_Z(Z)$. We saw that s extends to some open $U \supseteq Z$. Note that X is normal: two disjoint closed subsets have disjoint closed neighbourhoods. Thus we may produce $A \supseteq Z$ and $B \supseteq X\backslash U$ closed with $A \cap B = \varnothing$. By Urysohn's lemma, there exists an $f : X \to [0,1]$ such that $f|_A = 1$ and $f|_B = 0$. So $f \cdot s$ (the obvious extension) agrees on $\operatorname{int} A$ with s and hence extends by extends by 0 to a section $t \in C^0(X,\mathbb{R})$ lifting $s \in C^0(-,\mathbb{R})|_Z(Z)$.

(ii) Apply (i) on a compact neighborhood $Z \subseteq A$ and extend by 0 on $X\backslash A$. □

Using partitions of unity this result can be generalized to any C^k-manifold (where we assume manifolds to be paracompact).

Proposition 14.1.2 · *Let X be a C^k-manifold. Then $C^k(-,\mathbb{R})$ is soft.*

Lemma 14.1.3 · *Let X be a Hausdorff space and $\mathcal{O}_X$ a sheaf of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

(i) If X is paracompact and $\mathcal{O}_X$ is soft, then $\mathcal{F}$ is soft.

(ii) If X is locally compact and $\mathcal{O}_X$ is c-soft, then $\mathcal{F}$ is c-soft.

Proof. Let $Z \subseteq X$ be closed and $s \in \mathcal{F}|_Z(Z)$ and extend to $s \in \mathcal{F}(U)$ for $U \supseteq Z$ open (in (ii) assume both Z and $\overline{U}$ are compact). If $A = X\backslash U$ then $Z \coprod A \subseteq Z$ closed (and compact in case (ii)). The section $(1,0) \in \mathcal{O}_X(Z \coprod A) = \mathcal{O}_X(A) \times \mathcal{O}_X(Z)$ extends to $f \in \mathcal{O}_X(X)$ (compactly supported in case (ii)). Then $f \cdot s \in \mathcal{F}(U)$ extends by 0 to a (compactly supported) section $\mathcal{F}(X)$. □

is proof of (ii) correct?

Example 14.1.4 · On a manifold X the sheaf Ω^i_X of C^∞-sections of the cotangent bundle $\bigwedge^i T^*_X$ is a sheaf of $C^\infty(-,\mathbb{R})$ modules and hence Ω^i_X is soft and in particular acyclic. (We will use this for the comparison to de Rham cohomology).

14.2 The exponential sequence

Lemma 14.2.1 · *On any topological space X, the sequence $0 \to \underline{\mathbb{Z}} \to h_{\mathbb{R}} \to h_{S^1} \to 0$ is a short exact sequence of abelian sheaves.*

Proof. Exactness of $0 \to \underline{\mathbb{Z}} \to h_{\mathbb{R}} \to h_{S^1}$ is readily verified. Surjectivity of $h_{\mathbb{R}} \to h_{S^1}$ follows since any $U \to S^1$ locally lifts to $U_i \to \mathbb{R}$ since $\mathbb{R} \to S^1$ is a local homeomorphism. □

Remark 14.2.2 · Note that if X is simply connected, then any $U \to S^1$ already lifts to $U \to \mathbb{R}$.

'presumably in Munkres'

Lemma 14.2.3 · *If $X = \mathbb{R}$ or $[0,1]$ or $[0,1)$ then h_{S^1} is soft on X.*

Proof. Let $Z \subseteq X$ be closed and $s \in h_{S^1}|_Z(Z)$; it extends to $s \in h_{S^1}(U)$ for $U \supseteq Z$ open. But U is a disjoint union of contractible opens. So s lifts to $t_U \in h_{\mathbb{R}}(U)$. Since $h_{\mathbb{R}}$ is soft, we get $t \in h_{\mathbb{R}}(X)$ with $t|_U = t_U$ so its image in $h_{S^1}(X)$ does the job. □

Corollary 14.2.4 ·

(i) *If X is either $\mathbb{R}$, $[0,1]$ or $[0,1)$ then the cohomology of X with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ is*

$$H^i(X, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

(ii) *The compactly supported cohomology of $\mathbb{R}$, $[0,1]$ and $[0,1)$ with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ is:*

$$H^i_c(\mathbb{R}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 1, \\ 0 & \text{else,} \end{cases}$$

$$H^i_c([0,1], \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{else,} \end{cases}$$

$$H^i_c([0,1), \underline{\mathbb{Z}}) = 0 \qquad \text{for all } i.$$

Proof. (i) Lemma 14.2.1 gives a soft resolution of $\underline{\mathbb{Z}}$ that gives result.

(ii) For $X = [0,1]$, we have $H^i_c = H^i$ by Lemma 13.2.4. So by the lemma below, we obtain a long exact sequence

$$0 \longrightarrow H^0_c([0,1), \underline{\mathbb{Z}}) \longrightarrow H^0_c([0,1], \underline{\mathbb{Z}}) \longrightarrow H^0_c(\{1\}, \underline{\mathbb{Z}}) \longrightarrow H^1_c([0,1), \underline{\mathbb{Z}}) \longrightarrow H^1_c([0,1], \underline{\mathbb{Z}}) \longrightarrow H^1_c(\{1\}, \underline{\mathbb{Z}}) \longrightarrow \cdots$$

So since $H^i([0,1), \underline{\mathbb{Z}}) \xrightarrow{\sim} H^i_c(\{1\}, \underline{\mathbb{Z}})$ for all $i \geqslant 0$, $H^i_c([0,1], \underline{\mathbb{Z}}) = 0$. Similarly for $X = \mathbb{R} \cong (0,1)$, take $U = (0,1)$, $X = [0,1)$ and $Z = \{0\}$ (as in lemma below). We conclude that $H^i_c(\mathbb{R}, \underline{\mathbb{Z}}) = \mathbb{Z}$ and $H^i_c(\mathbb{R}, \underline{\mathbb{Z}}) = 0$ for all other i. □

For an alternative proof using the winding number, see [Ive86, Section III.4]. The following lemma was used in our proof.

Lemma 14.2.5 · *Let X be a locally compact Hausdorff topological space and $U \subseteq X$ open with complement Z. Then for any sheaf $\mathcal{F} \in \mathbf{Ab}(X)$ there is a long exact sequence,*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_c^0(U,\mathcal{F}|_U) & \longrightarrow & H_c^0(X,\mathcal{F}) & \longrightarrow & H_c^0(Z,\mathcal{F}|_Z) \\ & \hookrightarrow & H_c^1(U,\mathcal{F}|_U) & \longrightarrow & H_c^1(X,\mathcal{F}) & \longrightarrow & H_c^1(Z,\mathcal{F}|_Z) \\ & \hookrightarrow & \dots & & & & \end{array}$$

Proof. By Homework 3, Exercise 4, we have an exact sequence

$$0 \to j_!\,\mathcal{F}|_U \to \mathcal{F} \to i_*\,\mathcal{F}|_Z \to 0 \tag{14.1}$$

where $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ are the open and closed inclusions. By Additional exercise 13.1(d), we have $H_c^i(U, F|_U) = H_c^i(X, j_!\,F|_U)$ for all $i \in \mathbb{Z}_{\geqslant 0}$. Likewise, $H_c^i(Z,\mathcal{F}|_Z) = H_c^i(X, i_*\,\mathcal{F}|_Z)$. So applying the long exaxt sequence to 14.1 we get the desired result. □

strategy going forward

14.3 Higher pushforwards

> 'The fact that you can draw something doesn't mean it commutes – put that as a quote!'

Definition 14.3.1 · Let $f : Y \to X$ be a continuous map. Then the derived functors of the pushforward $f_* : \mathbf{Ab}(Y) \to \mathbf{Ab}(X)$ are called the *higher pushforwards* $\mathrm{R}^i f_*$.

Definition 14.3.2 · Write $\underline{H}^i : \mathbf{Ab}(X) \to \mathbf{PAb}(X)$ for the derived functors of the inclusion $\mathbf{PAb}(X) \hookrightarrow \mathbf{Ab}(X)$. Note that $(\underline{H}^i(\mathcal{F}))(U) = H^i(U,\mathcal{F})$: since the composite

$$\mathbf{Ab}(X) \hookrightarrow \mathbf{PAb}(X) \xrightarrow{\Gamma^{\mathrm{pre}}(U,-)} \mathbf{Ab}$$

is $\Gamma(U,-)$ and $\Gamma^{\mathrm{pre}}(U,-)$ is an exact functor this follows from Homework 7, Exercise 2(i).

Lemma 14.3.3 · *For a continuous map $f : Y \to X$ and an abelian sheaf $\mathcal{F}$, the sheaf $\mathrm{R}^i f_*\mathcal{F}$ is the sheafification of*

$$\mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Ab}, \quad U \mapsto H^i(f^{-1}(U),\mathcal{F}).$$

Proof. Observe that $U \mapsto H^i(f^{-1}(U),\mathcal{F})$ is the same as $f_*\underline{H}^i(\mathcal{F})$ where $f_* : \mathbf{PAb}(Y) \to \mathbf{PAb}(X)$. Consider the following commutative diagram (up to natural isomorphism):

$$\begin{array}{ccccc} \mathbf{Ab}(Y) & \xrightarrow{i_Y} & \mathbf{PAb}(Y) & & \\ {\scriptstyle f_*}\downarrow & \searrow^{F} & \downarrow{\scriptstyle f_*} & & \\ \mathbf{Ab}(X) & \xrightarrow[i_X]{} & \mathbf{PAb}(X) & \xrightarrow{(-)^\sharp} & \mathbf{Ab}(X) \end{array}$$

Note that $f_* : \mathbf{PAb}(Y) \to \mathbf{PAb}(X)$ is exact (since limits are computed objectwise, or since it has a left and a right adjoint). Also $(-)^\sharp$ is exact (see Example 10.1.2). The result now follows by two applications of Homework 7, Exercise 2. Firstly, since f_* is exact on $\mathbf{PAb}$, we have $f_*\underline{H}^i(\mathcal{F}) = \mathrm{R}^i F(\mathcal{F})$. Additionally, as the composition $(-)^\sharp \circ i_X = \mathrm{id}$ we have $(-)^\sharp \circ F = f_*$. So exactness of $(-)^\sharp$ gives $\mathrm{R}^i f_*\mathcal{F} = (\mathrm{R}^i F(\mathcal{F}))^\sharp = (f_*\underline{H}^i(\mathcal{F}))^\sharp$. □

Likewise, since $(-)^\sharp \circ i_X = \mathrm{id}$ and $(-)^\sharp$ is exact, we get

$$(\underline{H}^i(\mathcal{F}))^\sharp = \mathrm{R}^i((-)^\sharp \circ i_X)(\mathcal{F}) = \mathrm{R}^i \mathrm{id}(\mathcal{F}) = 0$$

if $i > 0$ since id is exact.

The following lemma expresses that the higher pushforward is the relative version of higher cohomology.

Lemma 14.3.4 · *Let $\mathcal{F}$ be an abelian sheaf on X and let $f : X \to *$ denote the unique map. Then $H^i(X, \mathcal{F}) = \mathrm{R}^i f_* \mathcal{F}$.*

Proof. Unravelling the definitions shows the equality holds for $i = 0$. It then follows for higher i. □

Exercise

Exercise 14.1 (The link between soft sheaves and partitions of unity) · Let $\mathcal{O}_X$ be a sheaf of rings on a Hausdorff space X.

(a) If X is paracompact, show that $\mathcal{O}_X$ is soft if and only if for every closed subset $Z \subseteq X$ and every open neighbourhood $U \supseteq Z$, there exists a section $f \in \mathcal{O}_X(X)$ such that $f\big|_Z = 1$ and $\mathrm{supp}(f) \subseteq U$.

(b) If X is locally compact, show that $\mathcal{O}_X$ is c-soft if and only if for every compact subset $Z \subseteq X$ and every open neighbourhood $U \supseteq Z$, there exists a section $f \in H^0_c(X, \mathcal{O}_X)$ such that $f\big|_Z = 1$ and $\mathrm{supp}(f) \subseteq U$.

LECTURE 15

Proper maps, proper pushforward, the proper base change theorem

We begin with some general theory of proper maps.

15.1 Proper maps

Definition 15.1.1 · A continuous map $f : Y \to X$ between topological spaces is *universally closed* if for any map $X' \to X$ the base change $X' \times_X Y \to Y$ is a closed map.

'The proof I'll give for this lemma is taken from Bourbaki. You'll hate it.'

Lemma 15.1.2 · *Let $X \in$ **Top**. The map $X \to *$ is universally closed if and only if X is compact.*

Proof. If X is compact, we need to show that the projection map $\pi : X \times Y \to Y$ is closed for any $Y \in$ **Top**. Let $Z \subseteq X \times Y$ closed and let $y \in Y \setminus \pi(Z)$. We will construct an open set around y that does not intersect $\pi(Z)$. Notice that $X \cong X \times \{y\} \subseteq Z^c$, so we can cover $X \times \{y\}$ by finitely many opens contained in Z^c. Their intersection is still an open open U, and the projection map is open so $\pi(U)$ is open containing y and not intersecting $\pi(Z)$. Conversely, assume $X \to *$ is universally closed. To prove compactness we will use prove that any collection of closed subsets of X with the finite intersection property has nonempty intersection. So let $Z_i \subseteq X$ ($i \in I$) be a such a collection. We use the notation $Z_J = \bigcap_{j\in J} Z_j$ for $J \subseteq I$. Define a new space

$$X' = X^{\text{disc}} \cup \{\infty\}$$

where $U \subseteq X'$ is open if and only if $\infty \in U$ implies $Z_J \subseteq U$ for some finite subset $J \subseteq I$. One checks this defines a topology on X'. Note that the subspace topology on X' is still the discrete topology. Furthermore, any open subset U containing ∞ contains some Z_J for J finite, and since $Z_J \neq \varnothing$ we have $U \cap X^{\text{disc}} \neq \varnothing$: X^{disc} is dense in X'. Let $\overline{\Delta}$ be the closure of $\{(x,x) \mid x \in X\} \subseteq X \times X'$. The image in X' of $\overline{\Delta}$ is closed by assumption and contains X so it is equal to X' because X is dense in X'. Thus there exists a point $(x, \infty) \in \overline{\Delta}$ for some $x \in X$. Now for all opens $U \subseteq X$ containing x and all $J \subseteq I$ finite, there exists a point $y \in X$ with $(y, y) \in U \times (Z_j \cup \{\infty\})$, and thus $x \in \overline{Z_J} = Z_J$ for all $J \subseteq I$ finite. ∎

why? Look at Remy's notes

Proposition 15.1.3 · *Let $f : Y \to X$ be a continuous map. The following are equivalent:*

(i) f is universally closed,

(ii) the product $f \times \mathrm{id}_W : Y \times W \to X \times W$ is closed for every space W,

(iii) f is closed and the fibre $f^{-1}(x)$ is compact for all $x \in X$,

(iv) f is closed and the fibre $f^{-1}(Z)$ is compact for all $Z \subseteq X$ closed.

Add the proof

Definition 15.1.4 · A map $f : Y \to X$ of topological spaces is called *separated* if the diagonal map $\Delta_{Y/X} : Y \to Y \times_X Y$ is closed. The map f is called *proper* if it is universally closed and separated.

Remark 15.1.5 · Some authors do not include the separated condition, which is automatic if X and Y are both locally compact Hausdorff spaces.

15.2 Proper pushforward

Definition 15.2.1 · Let $f : Y \to X$ be a map of topological spaces. The *proper pushforward* (sometimes called the *pushforward with proper support*) is the functor $f_! : \mathbf{Ab}(Y) \to \mathbf{Ab}(X)$ defined, given a sheaf $\mathcal{F} \in \mathbf{Ab}(Y)$, by

$$(f_!\mathcal{F})(U) = \{ s \in \mathcal{F}(f^{-1}(U)) \mid f\big|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to U \text{ is proper} \}.$$

Remark 15.2.2 · In Homework 8 you are asked to show this assignment gives a sheaf on X.

Example 15.2.3 · In Additional exercise 15.3 you are asked to show that for the inclusion of an open subset $j : U \hookrightarrow X$ the proper pushforward becomes the extension-by-zero functor.

We have a functor $f_!$ between abelian categories. One checks it is left exact.

Definition 15.2.4 · The ith *higher pushforward with proper support* is the ith right derived functor $\mathrm{R}^i f_! : \mathbf{Ab}(Y) \to \mathbf{Ab}(X)$ of the *proper pushforward*.

Example 15.2.5 · If $j : U \hookrightarrow X$ is an open subset then $\mathrm{R}^i j_! = 0$ for all $i > 0$ since $j_!$ is exact.

Exercise 15.2.6 · If X is Hausdorff then $H^i_c(X, \mathcal{F}) = \mathrm{R}^i f_! \mathcal{F}$ for any sheaf $\mathcal{F} \in \mathbf{Ab}(X)$ and $f : X \to *$.

15.3 The proper base change theorem

'We can't prove this because we're dumb. Which is fine.'

We are now ready to state one of the main results of the course. We do so in two versions, one involving the pushforward and one involving the proper pushforward.

Theorem 15.3.1 · *Let $f : Y \to X$ be a proper map of topological spaces and assume that X and Y are either both paracompact Hausdorff or both locally compact Hausdorff. For $\mathcal{F} \in \mathbf{Ab}(Y)$ the natural map*

$$(\mathrm{R}^i f_* \mathcal{F})_x \to H^i(f^{-1}(x), \mathcal{F})$$

is an isomorphism for all $x \in X$ and for all $i \geqslant 0$.

Remark 15.3.2 · Theorem 15.3.1 is true for f proper with no further hypotheses, but we will not prove it in this course. See [Sta24, Lemma 09V6] for a proof of the general case.

Theorem 15.3.3 · *Let $f : Y \to X$ be a continuous map of locally compact Hausdorff spaces. For any $\mathcal{F} \in \mathbf{Ab}(Y)$ the natural map*

$$(\mathrm{R}^i f_! \mathcal{F})_x \to H^i_c(f^{-1}(x), \mathcal{F})$$

is an isomorphism for all $x \in X$ and all $i \geqslant 0$.

We will first prove Theorem 15.3.1. We will need the following lemma.

Lemma 15.3.4 · *Let $i : Z \hookrightarrow X$ be a closed subset. Assume that either*

(i) *X is paracompact Hausdorff or*

(ii) *X is locally compact Hausdorff and that Z is compact.*

Then there is an isomorphism

$$\operatorname*{colim}_{Z\subseteq U \text{ open}} H^i(U,\mathcal{F}) \xrightarrow{\sim} H^i(Z,\mathcal{F})$$

for all sheaves and all $i \geqslant 0$.

Proof. We proved in Proposition 13.3.3 that the statement holds for $i = 0$. Note that the pullback $i^* : \mathbf{Ab}(X) \to \mathbf{Ab}(Z)$ preserves soft sheaves if X is paracompact Hausdorff (respectively c-soft sheaves if X is locally compact Hausdorff, irrespective of whether Z is compact): if $W \subseteq Z$ is closed, then any $s \in \mathcal{F}(W)$ extends to $\mathcal{F}(X)$ hence also to $\mathcal{F}(Z)$ (note that W is compact if Z is). Now given an injective resolution $0 \to \mathcal{F} \to I^\bullet$, it is (c)-soft, and $i^*I^\bullet$ is a (c)-soft resolution of $i^*\mathcal{F}$. Unravelling the definitions, we obtain the desired result:

where have we used the assumptions on X?

$$\begin{aligned} H^i(Z,\mathcal{F}|_Z) &= H^i(\Gamma(Z,i^*I^\bullet)) \\ &\cong H^i(\varinjlim_{Z\subseteq U}\Gamma(U,i^*I^\bullet)) \\ &\cong \varinjlim_{Z\subseteq U} H^i(\Gamma(U,i^*I^\bullet)) \\ &= \varinjlim_{Z\subseteq U} H^i(U,\mathcal{F}). \end{aligned}$$

□

We are now ready to prove the first version of the Proper Base Change Theorem.

Proof (of Theorem 15.3.1). Recall that $\mathrm{R}^i f_*\mathcal{F}$ is the sheafification of the assignment $U \mapsto H^i(f^{-1}(U),\mathcal{F})$, so that

$$(\mathrm{R}^i f_*\mathcal{F})_x = \operatorname*{colim}_{x\in U \text{ open}} H^i(f^{-1}(U),\mathcal{F}).$$

By Lemma 15.3.4 we know that

$$H^i(f^{-1}(x),\mathcal{F}) = \operatorname*{colim}_{f^{-1}(x)\subseteq V \text{ open}} H^i(V,\mathcal{F}),$$

since f^{-1} is compact. By properness, the opens $f^{-1}(U)$ are coinitial amongst the $V \supset f^{-1}(x)$: an open V contains $f^{-1}(f(V^C)^C)$. So the colimits agree. □

Clarify argument at the end.

Lemma 15.3.5 · *Let $i : Z \hookrightarrow X$ be a closed subspace of a locally compact Hausdorff space. Let $\mathcal{F} \in \mathbf{Ab}(X)$ be a c-soft sheaf of abelian groups. Then $i^*\mathcal{F}$ is c-soft and the map*

$$H^0_c(X,\mathcal{F}) \to H^0_c(Z,\mathcal{F})$$

is surjective.

Proof. We already proved in Lemma 15.3.4 that $i^*\mathcal{F}$ is c-soft. Given $s \in H^0_c(Z,\mathcal{F})$, choose a compact neighborhood $K \subseteq X$ of $\sup(s)$. The section $s|_{Z\cap K}$ extends to a section

$$s_1 \in H^0((Z \cap K) \cup \partial K,\mathcal{F})$$

that is 0 on the boundary ∂K. This section further extends to $s_2 \in H^0(K,\mathcal{F})$ by c-softness. We can extend by 0 to a section in $H^0_c(X,\mathcal{F})$, since $s_2|_{\partial K} = 0$. □

Add details and verify.

Proof (of Theorem 15.3.3). We have a map

$$(f_!\mathcal{F})_x = \varinjlim_{x\in U \text{ open}} (f_!\mathcal{F})(U) \xrightarrow{\varphi} H^0_c(f^{-1}(x),\mathcal{F})$$

given by $s \mapsto s|_{f^{-1}(x)}$ which we claim is injective. If $s \in (f_!\mathcal{F})(U)$ maps to zero under this map, then

$$\sup(s) \cap f^{-1}(x) = \varnothing.$$

Since the map $\sup(s) \to U$ is proper, the set $V = U \setminus f(\sup(s))$ is an open neighborhood of x with the property that $\sup(s) \cap f^{-1}(V) = \varnothing$. Thus s maps to zero in $(f_!\mathcal{F})(V)$, proving injectivity of φ. If $\mathcal{F}$ is c-soft, then the composition

$$H^0_c(Y, \mathcal{F}) \hookrightarrow (f_!\mathcal{F})(X) \to (f_!\mathcal{F})_x \xrightarrow{\varphi} H^0_c(f^{-1}(x), \mathcal{F})$$

is surjective by Lemma 15.3.5 so the map φ is an isomorphism.

Now take an injective resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots$$

Then $i^*I^\bullet$ is a c-soft resolution of $i^*\mathcal{F}$ by Lemma 15.3.5, so we have

$$H^i_c(f^{-1}(x), \mathcal{F}) = H^i(H^0_c(f^{-1}(x), I^\bullet)) = H^i(\varinjlim_{x \in U \text{ open}} (f_!I^\bullet(U))) = \varinjlim_{x\in U} H^i((f_!I^\bullet)(U)) = (\mathrm{R}^i f_!\mathcal{F})_x,$$

and we are done. ☐ expand details, verify

The next corollary explains the use of *base change* in the namings of Theorem 15.3.1 and Theorem 15.3.3.

Corollary 15.3.6 · *Let the diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

be a pullback square in **Top**.

(i) If f is a proper map and X and X' are either both paracompact Hausdorff or both locally compact Hausdorff, then there are canonical isomorphisms

$$g^*\mathrm{R}^i f_* \xrightarrow{\cong} (\mathrm{R}^i f'_*)g'^* : \mathbf{Ab}(Y) \to \mathbf{Ab}(X).$$

(ii) If X, X' and Y' are locally compact Hausdorff, then there are canonical isomorphisms

$$g^*\mathrm{R}^i f_! \xrightarrow{\cong} (\mathrm{R}^i f'_!)g'^* : \mathbf{Ab}(Y) \to \mathbf{Ab}(X').$$

Proof (sketch). ☐ write

LECTURE 16

Homotopy invariance, Čech cohomology

16.1 Homotopy invariance

> 'It's just annoying homological algebra, and this is slightly less annoying homological algebra.'

In this section, we prove that sheaf cohomology is homotopy invariant: we will show that homotopic maps $f : X \to Y$ induce the same map $H^i(X, \underline{A}) \to H^i(Y, \underline{A})$ on cohomology with coefficients in the constant sheaf on some abelian group A. From this, it follows that homotopy equivalences induce isomorphisms in cohomology, whence we obtain in particular a computation of the sheaf cohomology with constant coefficients of all contractible spaces. Although homotopy invariance holds for continuous maps between arbitrary topological spaces, we will restrict to paracompact Hausdorff or locally compact spaces, because we have only developed the machinery for this class of spaces.

The first lemma of today shows that compactly supported sheaf cohomology preserves filtered colimits. We could have presented this proof right after defining compactly support cohomology; it does not use later results.

Lemma 16.1.1 · *If X is a locally compact Hausdorff space, then the compactly supported cohomology $H^i_c(X, -) :$ $\mathbf{Ab}(X) \to \mathbf{Ab}$ preserves filtered colimits.*

Proof. Consider the one-point compactification $j : X \hookrightarrow \overline{X}$ of X. For any sheaf $\mathcal{F}$, we have

$$H^i_c(X, \mathcal{F}) = H^i_c(\overline{X}, j_!\mathcal{F}) = H^i(\overline{X}, j_!\mathcal{F})$$

by Additional exercise 13.1(d) and since $\overline{X}$ is compact. Note that $j_!$ preserves filtered colimits, ref result
for instance because it is the left adjoint of the pullback j^* (Additional exercise 16.1). Thus we may assume X is compact.

Now let $\mathcal{F}_- : \mathcal{I} \to \mathbf{Ab}(X)$ be a filtered diagram of sheaves. For $i = 0$, there is a canonical map

$$\operatorname*{colim}_{i \in \mathcal{I}} \mathcal{F}_i(X) \to (\operatorname*{colim}_{i \in \mathcal{I}} \mathcal{F}_i)(X).$$

We will show that this is an isomorphism.

For injectivity, if a section $s \in \mathcal{F}_i(X)$ maps to zero, then there exists a finite cover $X = U_1 \cup \cdots \cup U_n$ (by compactness of X) such that $s|_{U_k}$ is zero in $\operatorname{colim}_{i \in \mathcal{I}} \mathcal{F}_i(U_k)$ for all $k \in \{1, \dots, n\}$, so there is some arrow $i \to j$ in $\mathcal{I}$ such that $s|_{U_k}$ is zero in $\mathcal{F}_j(U_k)$ for all k (since $\mathcal{I}$ is filtered). Then s becomes zero in $\mathcal{F}_j(X)$; so the kernel of the map is zero, whence it is injective.

For surjectivity, let $s \in (\operatorname{colim}_{i \in \mathcal{I}} \mathcal{F}_i)(X)$ and choose an open cover $X = U_1 \cup \cdots \cup U_n$ and $i \in \mathcal{I}$ such that $s|_{U_k}$ comes from $t_k \in \mathcal{F}_i(U_k)$ for all k (using compactness of X and since the sheaf colimit is the sheafification of the presheaf colimit, Theorem 7.4.1). Using Lemma 13.3.2, choose an open cover $X = V_1 \cup \cdots \cup V_n$ such that $\overline{V_k} \subseteq U_k$ for all k. Note that $(-)|_{\overline{V_k}}$ preserves colimits (it is the pullback i_k^* along the closed inclusion $i_k : \overline{V_k} \hookrightarrow X$, which is right adjoint to the pushforward $(i_k)_*$ by Proposition 5.2.1). Thus, since the map

$$\operatorname*{colim}_{i \in \mathcal{I}} \mathcal{F}_i(\overline{V_k} \cap \overline{V_\ell}) \to (\operatorname*{colim}_{i \in \mathcal{I}} \mathcal{F}_i)(\overline{V_k} \cap \overline{V_\ell})$$

is injective by the above, there is an arrow $i \to j$ in $\mathcal{I}$ such that $t_k|_{\overline{V_k} \cap \overline{V_\ell}} - t_\ell|_{\overline{V_k} \cap \overline{V_\ell}}$ maps to zero in $\mathcal{F}_j(\overline{V_k} \cap \overline{V_\ell})$ for all $k, \ell \in \{1, \dots, n\}$. Then the sections $(t_k|_{V_k})_{k=0}^n$ glue to a section in $\mathcal{F}_j(X)$ lifting s, proving the result for $i = 0$.

This also shows that filtered colimits of soft sheaves are soft. So if

$$\mathcal{F} \mapsto (0 \to \mathcal{F} \to \mathcal{S}_0(\mathcal{F}) \to \mathcal{S}_1(\mathcal{F}) \to \dots)$$

is a functorial soft resolution (such as the Godement resolution of Remark 13.1.8), then

$$0 \to \operatorname*{colim}_{i \in \mathcal{I}} \mathcal{F}_i \to \operatorname*{colim}_{i \in \mathcal{I}} \mathcal{S}_0(\mathcal{F}_i) \to \operatorname*{colim}_{i \in \mathcal{I}} \mathcal{S}_1(\mathcal{F}_i) \to \dots$$

is a soft resolution, so we win by the $i = 0$ case. □

Corollary 16.1.2 · *If X is a locally compact Hausdorff space with compactly supported cohomology with $\underline{\mathbb{Z}}$ coefficients given by*

$$H_c^i(X, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

then we have

$$H_c^i(X, \underline{A}) = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

for any abelian group A.

Example 16.1.3 · We saw in Corollary 14.2.4 that the hypothesis of Corollary 16.1.2 is satisfied by $[0, 1]$. In Homework 8, Exercise 4(b) you show that $[0, 1]^n$ satisfies the hypothesis for all $n \geqslant 0$.

Proof (of Corollary 16.1.2). If A is finitely generated, say by a short exact sequence

$$0 \to \mathbb{Z}^r \to \mathbb{Z}^s \to A \to 0,$$

then exactness of

$$0 \to \underline{\mathbb{Z}^r} \to \underline{\mathbb{Z}^s} \to \underline{A} \to 0$$

by Lemma 9.1.1 gives exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_c^0(X, \underline{\mathbb{Z}^r}) & \longrightarrow & H_c^0(X, \underline{\mathbb{Z}^r}) & \longrightarrow & H_c^0(X, \underline{A}) & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathbb{Z}^r & \longrightarrow & \mathbb{Z}^s & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

and

$$\underbrace{H_c^i(X, \underline{\mathbb{Z}^s})}_{=0} \to H_c^i(X, \underline{A}) \to \underbrace{H_c^{i+1}(X, \underline{\mathbb{Z}^r})}_{=0}$$

for $i > 0$. This proves the result for finitely generated A.

For general abelian groups A, write A as the filtered colimit $A = \operatorname{colim}_{B \subseteq A \text{ f.g.}} B$ of all finitely-generated subgroups B and use that $A \mapsto \underline{A}$ preserves colimits (again Lemma 9.1.1). □

Theorem 16.1.4 (Vietoris–Begle mapping theorem) · *Let $f : Y \to X$ be a proper map such that*

$$H^i(f^{-1}(x), \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases}$$

for all $x \in X$. Then for all abelian sheaves $\mathcal{F}$ on X, the maps

$$H^i(X, \mathcal{F}) \to H^i(Y, f^*\mathcal{F})$$

are isomorphisms for all $i \in \mathbb{N}$.

Again, we will only prove this when X is paracompact Hausdorff or locally compact Hausdorff.

For the statement of the theorem to make sense, we still need to define the maps; we will do this in a more general context. Let

$$\mathcal{A} \underset{F}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

be left exact functors between abelian categories where $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. (Since L is a left adjoint, it will also be exact.) We proved in Lemma 11.2.5 that F preserves injective objects. For an object B of $\mathcal{B}$, we will construct maps

$$\mathrm{R}^i G(B) \to \mathrm{R}^i(GF)(LB)$$

for all i. To obtain the maps from the theorem, we apply this to the functors

$$\mathbf{Ab}(Y) \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Ab}(X) \xrightarrow{\Gamma(X,-)} \mathbf{Ab}$$

Choose injective resolutions

$$0 \to B \to I^0 \to I^1 \to \dots, \quad 0 \to LB \to J^0 \to J^1 \to \dots.$$

Exactness of L gives an exact sequence

$$0 \to LB \to LI^0 \to LI^1 \to \dots,$$

but this need not be a G-acyclic resolution, so we cannot use this to compute derived functors. We saw in Corollary 12.2.6 that we can extend the identity of LB to a chain map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & LB & \longrightarrow & LI^0 & \longrightarrow & LI^1 & \longrightarrow & \dots \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & LB & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \dots \end{array}$$

which is unique up to homotopy. The desired map can now be defined as the composite

$$\mathrm{R}^i G(B) = H^i(G(I^\bullet)) \xrightarrow{\eta} H^i(GF(LI^\bullet)) \to H^i(GF(J^\bullet)) = \mathrm{R}^i(GF)(LB)$$

where η is the unit of the adjunction $L \dashv F$.

Exercise 16.1.5 · Show that the above construction is independent of choices.

Remark 16.1.6 · One can even construct maps

$$\mathrm{R}^i G(FA) \to \mathrm{R}^i(GF)(A)$$

for all objects A of $\mathcal{A}$, without assuming the existence of L but assuming F takes injective objects to G-acyclic objects, and then the above map coincides with the composite

$$\mathrm{R}^i G(B) \to \mathrm{R}^i G(FLB) \to \mathrm{R}^i(GF)(LB)$$

where the first map is applying the unit only to B.

Proof (of Theorem 16.1.4). The unit $\mathcal{F} \to f_* f^* \mathcal{F}$ of the pushforward–pullback adjunction $f^* \dashv f_*$ is an isomorphism for every sheaf $\mathcal{F}$ on X: for each $x \in X$, the stalk of this map is

$$\mathcal{F}_x \to (f_* f^* \mathcal{F})_x = H^0(f^{-1}(x), f^* \mathcal{F})$$

by Theorem 15.3.1. But $(f^*\mathcal{F})|_{f^{-1}(x)} = \underline{\mathcal{F}_x}$ by commutativity of the diagram

$$\begin{array}{ccc} f^{-1}(x) & \longrightarrow & Y \\ \big\downarrow & \lrcorner & \big\downarrow f \\ * & \xrightarrow[x]{} & X \end{array}$$

since $\underline{\mathcal{F}_x}$ is the pullback along $x : * \to X$. Corollary 16.1.2 shows that

$$H^i(f^{-1}(x), f^*\mathcal{F}) = \begin{cases} \mathcal{F}_x & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

so $\mathcal{F}_x \to (f_* f^* \mathcal{F})_x$ is an isomorphism. Likewise, $\mathrm{R}^i f_*(f^*\mathcal{F})_x = H^i(f^{-1}(x), f^*\mathcal{F}) = 0$ for $i > 0$. Then Homework 7, Exercise 2 shows that

$$H^i(Y, f^*\mathcal{F}) = H^i(X, f_* f^* \mathcal{F}) = H^i(X, \mathcal{F}). \qquad \square$$

Corollary 16.1.7 · *If $f, g : Y \to X$ are homotopic maps, then the maps*

$$f^*, g^* : H^i(X, \underline{A}) \to H^i(Y, \underline{A})$$

agree for all $i \geqslant 0$ and all abelian groups A.

Once again, we will prove this under the assumption that X and Y are paracompact Hausdorff or locally compact Hausdorff.

Proof. Consider the diagram

$$H^i(Y, \underline{A}) \xrightarrow[\cong]{\mathrm{pr}_Y^*} H^i(Y \times [0,1], \underline{A}) \underset{1^*}{\overset{0^*}{\rightrightarrows}} H^i(Y, \underline{A})$$

in which pr_Y^* is an isomorphism by Theorem 16.1.4 and in which both composites are the identity. Then $0^* = 1^*$ (both being the inverse of pr_Y^*), so if $h : Y \times [0,1] \to X$ is a homotopy from f to g, then

$$f^* = 0^* \circ h^* = 1^* \circ h^* = g^*. \qquad \square$$

Corollary 16.1.8 · *If $f : X \to Y$ is a homotopy equivalence (of paracompact Hausdorff or locally compact Hausdorff spaces), then*

$$f^* : H^i(X, \underline{A}) \xrightarrow{\cong} H^i(Y, \underline{A})$$

is an isomorphism for any abelian group A. In particular, if X and Y are homotopy equivalent, then $H^i(X, \underline{A}) \cong H^i(Y, \underline{A})$ for any A.

Proof. If $g : Y \to X$ is a homotopy inverse of f, then

$$g^* \circ f^* = (gf)^* = \mathrm{id}^* = \mathrm{id}$$

by Corollary 16.1.7 and similarly $f^* \circ g^* = \mathrm{id}$, so f^* is an isomorphism. $\square$

Corollary 16.1.9 · *If X is contractible (and paracompact Hausdorff or locally compact Hausdorff), then*

$$H^i(X, \underline{A}) = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases}$$

for any abelian group A.

Proof. Directly from Corollary 16.1.8 and the computation of $H^i(*, \underline{A})$ (for example by Corollary 16.1.2). □

Example 16.1.10 · We can finally compute that

$$H^i(\mathbb{R}^n, \underline{A}) = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Next week, we will compute the sheaf cohomology of $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$.

16.2 Čech cohomology

Notation 16.2.1 · If $\{U_i \hookrightarrow X\}_{i\in I}$ is an open cover of a space X, we write

$$U_{i_0,\dots,i_n} := U_{i_0} \cap \dots \cap U_{i_n}$$

for $i_0, \dots, i_n \in I$. If $J \subseteq I$ is a subset, we write

$$U_J := \bigcap_{j\in J} U_j.$$

Definition 16.2.2 · Let X be a topological space and let $\mathcal{U} = \{U_i \hookrightarrow X\}_{i\in I}$ be an open cover of X. The *(alternating) Čech complex* of an abelian presheaf F on X with respect to $\mathcal{U}$ is the cochain complex

$$0 \to \check{C}^0(\mathcal{U}, F) \xrightarrow{d^0} \check{C}^1(\mathcal{U}, F) \xrightarrow{d^1} \check{C}^2(\mathcal{U}, F) \to \dots$$

where the cochains in degree n are given by

$$\check{C}^n(\mathcal{U}, F) := \left\{ (s_{i_0,\dots,i_n})_{(i_0,\dots,i_n)\in I^{n+1}} \in \prod_{(i_0,\dots,i_n)\in I^{n+1}} F(U_{i_0,\dots,i_n}) \;\middle|\; \begin{array}{c} s_{i_0,\dots,i_n} = 0 \text{ if } \#\{i_0,\dots,i_n\} \leqslant n \\ s_{\sigma(i_0),\dots,\sigma(i_n)} = \operatorname{sgn}(\sigma)\cdot s_{i_0,\dots,i_n} \text{ for } \sigma \in S_{n+1} \end{array} \right\}$$

with differential $d^n : \check{C}^n(\mathcal{U}, F) \to \check{C}^{n+1}(\mathcal{U}, F)$ given by

$$(s_{i_0,\dots,i_n})_{(i_0,\dots,i_n)\in I^{n+1}} \mapsto \Big(\sum_{j=0}^{n+1} (-1)^j s_{i_0,\dots,i_{j-1},i_{j+1},\dots,i_{n+1}}|_{U_{i_0,\dots,i_{n+1}}}\Big)_{(i_0,\dots,i_{n+1})\in I^{n+2}}.$$

The *Čech cohomology* $\check{H}^\bullet(\mathcal{U}, F)$ of F with respect to $\mathcal{U}$ is the cohomology of the Čech complex:

$$\check{H}^i(\mathcal{U}, F) := H^i(\check{C}^\bullet(\mathcal{U}, F)).$$

Exercise 16.2.3 · Show that the Čech complex is a cochain complex, that is, $d^{n+1}d^n = 0$ for all n.

There are some variants of the Čech complex:

- Instead of alternating cochains, one can take

$$\check{C}^n_{\text{full}}(\mathcal{U}, F) := \prod_{(i_0,\dots,i_n)\in I^{n+1}} F(U_{i_0,\dots,i_n}).$$

- The *choice* of a linear order on I gives an isomorphism

$$\check{C}^n(\mathcal{U}, F) \cong \prod_{i_0<\dots<i_n} F(U_{i_0,\dots,i_n}) =: \check{C}^n_{\text{ord}}(\mathcal{U}, F).$$

These complexes are related by maps

$$\check{C}^\bullet(\mathcal{U},F) \hookrightarrow \check{C}^\bullet_{\mathrm{full}}(\mathcal{U},F) \twoheadrightarrow \check{C}^\bullet_{\mathrm{ord}}(\mathcal{U},F), \qquad \check{C}^\bullet(\mathcal{U},F) \xrightarrow{\cong} \check{C}^\bullet_{\mathrm{ord}}(\mathcal{U},F)$$

One can show that these maps are chain homotopy equivalences – so in particular quasi-isomorphisms, whence the complexes have the same cohomology –, but this is annoying combinatorics; see [Con].

Example 16.2.4 · If $\mathcal{U} = \{U_1 \hookrightarrow X, U_2 \hookrightarrow X\}$, the ordered Čech complex is

$$\begin{aligned} 0 \longrightarrow F(U_1) \times F(U_2) &\xrightarrow{d^0} F(U_1 \cap U_2) \xrightarrow{d^1} 0 \longrightarrow \dots \\ (s_1, s_2) &\longmapsto s_2|_{U_1 \cap U_2} - s_1|_{U_1 \cap U_2} \end{aligned}$$

Example 16.2.5 · If $\mathcal{U} = \{U_1 \hookrightarrow X, U_2 \hookrightarrow X, U_3 \hookrightarrow X\}$, the ordered Čech complex is

$$0 \longrightarrow F(U_1) \times F(U_2) \times F(U_3) \xrightarrow{d^0} F(U_1 \cap U_2) \times F(U_1 \cap U_3) \times F(U_2 \cap U_3) \xrightarrow{d^1} F(U_1 \cap U_2 \cap U_3) \xrightarrow{d^2} 0 \longrightarrow \dots$$

$$(s_1, s_2, s_3) \longmapsto (s_2 - s_1, s_3 - s_1, s_3 - s_2)$$

$$(t_{12}, t_{13}, t_{23}) \longmapsto t_{23} - t_{13} + t_{12}$$

(Here we notationally suppress the restrictions.)

Remark 16.2.6 · There is a map

$$F(X) \to \check{C}^0(\mathcal{U}), \quad s \mapsto (s|_{U_i})_{i \in I}$$

and the composition $F(X) \to \check{C}^0(\mathcal{U}, F) \to \check{C}^1(\mathcal{U}, F)$ is zero. The sheaf condition says that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is an isomorphism for all $U \subseteq X$ open and all open covers $\mathcal{U}$ of U when $\mathcal{F}$ is a sheaf.

Next week, we will show that $\check{H}^i(\mathcal{U}, -)$ is the ith right derived functor $\mathrm{R}^i\check{H}^0(\mathcal{U}, -)$ as functors $\mathbf{PAb}(X) \to \mathbf{Ab}$.

LECTURE 17

Čech cohomology, comparison with sheaf cohomology

17.1 Čech cohomology

'Since I don't know what I'm doing,
let me take a break.'

Recall from the last lecture that we can define the cochains in degree n of the Čech complex of an abelian presheaf F on a space X with respect to an open cover $\mathcal{U} = \{U_i \hookrightarrow X\}_{i \in I}$ as

$$\check{C}^n(\mathcal{U}, F) \cong \prod_{i_0 < \ldots < i_n} \mathcal{F}(U_{i_0,\ldots,i_n})$$

if the indexing set I is totally ordered. Throughout this lecture, we work with this definition, so we assume I is totally ordered; we can also do without this assumption, but prefer the notational simplicity we obtain with this assumption.

The Čech cohomology, which is the cohomology of the Čech complex, is defined for all presheaves, not just sheaves. Today, we will discuss some results about Čech cohomology which do hold on the category of presheaves, but not on the full subcategory of sheaves. In the following lemma we already see why this is important.

Lemma 17.1.1 · *The functor $\check{H}^i(\mathcal{U}, -) : \mathbf{PAb}(X) \to \mathbf{Ab}$ define a δ-functor.* define

Proof. If $0 \to F \to G \to H \to 0$ is a short exact sequence of presheaves, then for all $i_0 < \ldots < i_n$, the sequence

$$0 \to F(U_{i_0,\ldots,i_n}) \to G(U_{i_0,\ldots,i_n}) \to H(U_{i_0,\ldots,i_n}) \to 0$$

is exact (but this is not true in the category of sheaves!). Thus, the sequence

$$0 \to \check{C}^\bullet(\mathcal{U}, F) \to \check{C}^\bullet(\mathcal{U}, G) \to \check{C}^\bullet(\mathcal{U}, H) \to 0$$

of complexes is exact, giving the required long exact sequence. □

Theorem 17.1.2 · *If I is an injective abelian presheaf on a space X, then the Čech cohomology $\check{H}^i(\mathcal{U}, I)$ vanishes for all $i > 0$.*

To prove this theorem, we introduce some notation and prove a lemma.

Notation 17.1.3 · Let $K^\bullet$ be the cochain complex of presheaves

$$\ldots \to K^{-2} \to K^{-1} \to K^0 \to 0 \to \ldots$$

given by

$$K^{-n} := \bigoplus_{i_0 < \ldots < i_n} \underline{\mathbb{Z}}^{\text{pre}}_{U_{i_0,\ldots,i_n}}$$

with differential $d^{-n} : K^{-n} \to K^{-n+1}$ given on the factor $\underline{\mathbb{Z}}^{\mathrm{pre}}_{U_{i_0,\dots,i_n}}$ by

$$\sum_{k=0}^{n} (-1)^k \left(\underline{\mathbb{Z}}^{\mathrm{pre}}_{U_{i_0,\dots,i_n}} \to \underline{\mathbb{Z}}^{\mathrm{pre}}_{U_{i_0,\dots,i_{k-1},i_{k+1},\dots,i_n}}\right) : \underline{\mathbb{Z}}^{\mathrm{pre}}_{U_{i_0,\dots,i_n}} \to K^{-(n-1)} = \bigoplus_{j_0<\dots<j_{n-1}} \underline{\mathbb{Z}}^{\mathrm{pre}}_{U_{j_0,\dots,j_{n-1}}}$$

of the maps induced by the inclusions $U_{i_0,\dots,i_n} \hookrightarrow U_{i_0,\dots,i_{k-1},i_{k+1},\dots,i_n}$. Then we have $\mathrm{Hom}_{\mathbf{PAb}(X)}(K^{-\bullet}, F) \cong \check{C}^{\bullet}(\mathcal{U}, F)$, more or less by definition.

Lemma 17.1.4 · *Let $\underline{\mathbb{Z}}^{\mathrm{pre}}_{\mathcal{U}} \subseteq \underline{\mathbb{Z}}^{\mathrm{pre}}_{X}$ be the image of $K^0 = \bigoplus_{i\in I} \underline{\mathbb{Z}}^{\mathrm{pre}}_{U_i} \to \underline{\mathbb{Z}}^{\mathrm{pre}}_{X}$, whose value on an open V is $\mathbb{Z}$ if $V \subseteq U_i$ for some i, and zero otherwise. Then the sequence*

$$K^{\bullet}_{+} = (\dots \to K^{-2} \to K^{-1} \to K^{0} \to \underline{\mathbb{Z}}^{\mathrm{pre}}_{\mathcal{U}} \to 0 \to \dots)$$

is exact in **PAb**(X). *In other words, the cohomology of $K^{\bullet}$ is*

$$H^i(K^{\bullet}) = \begin{cases} \underline{\mathbb{Z}}^{\mathrm{pre}}_{\mathcal{U}} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Proof. □ write

Proof (of Theorem 17.1.2). We saw that $\check{C}^{\bullet}_{\mathrm{ord}}(\mathcal{U}, I) = \mathrm{Hom}_{\mathbf{PAb}(X)}(K^{\bullet}, I)$, and $K^{\bullet}_{+}$ is exact. Since $\mathrm{Hom}(-, I)$ is exact (because I is injective), we conclude that the sequence

$$0 \to \mathrm{Hom}(\underline{\mathbb{Z}}^{\mathrm{pre}}_{\mathcal{U}}, I) \to \check{C}^0(\mathcal{U}, I) \to \check{C}^1(\mathcal{U}, I) \to \dots$$

is exact, so $\check{H}^i(\mathcal{U}, I) = 0$ for $i > 0$. □

Corollary 17.1.5 · *The δ-functor $\check{H}^i(\mathcal{U}, -) :$ **PAb**$(X) \to$ **Ab** *is isomorphic as a δ-functor to* $\mathrm{R}^i\check{H}^0(\mathcal{U}, -)$.

The proof of this corollary is Additional exercise 16.2(c).

Remark 17.1.6 · The higher Čech cohomology functors are derived functors from the category of presheaves, not the category of sheaves. In fact, the composite

$$\mathbf{Ab}(X) \hookrightarrow \mathbf{PAb}(X) \xrightarrow{\check{H}^0(\mathcal{U}, -)} \mathbf{Ab}$$

is the global sections functor $\Gamma(X, -)$.

write

Corollary 17.1.7 · *Let X be a topological space with an open cover $\mathcal{U} = \{U_i \hookrightarrow X\}_{i\in I}$ and let $\mathcal{F}$ be an abelian sheaf on X. If $H^i(U_{i_0,\dots,i_n}, \mathcal{F}) = 0$ for all $i > 0$, all $n \geqslant 0$ and all $i_0, \dots, i_n \in I$, then the map*

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$$

is an isomorphism.

Proof. □ write

Example 17.1.8 · If all intersections $U_{i_0,\dots,i_n}$ are disjoint unions of contractibles, then $H^i(U_{i_0,\dots,i_n}, \underline{\mathbb{Z}}) = 0$ for all $i > 0$, so $\check{H}^i(\mathcal{U}, \underline{\mathbb{Z}}) = H^i(X, \underline{\mathbb{Z}})$.

Example 17.1.9 · We will compute the cohomology $H^i(S^1, \underline{\mathbb{Z}})$ of the circle again. Let x_1 and x_2 denote the 'north pole' and 'south pole' of the circle, and define $U_1 := S^1 \setminus \{x_2\}$ and $U_2 := S^1 \setminus \{x_1\}$; write $\mathcal{U}$ for the open cover $\{U_1 \hookrightarrow S^1, U_2 \hookrightarrow S^1\}$. The cohomology of the circle can be computed with the Mayer–Vietoris sequence associated to this cover, but we can also use Čech cohomology: since $U_1 \cong U_2 \cong \mathbb{R}$ and $U_1 \cap U_2 \cong \mathbb{R} \sqcup \mathbb{R}$, we have picture

$H^i(U_{i_0,\dots,i_n}, \underline{\mathbb{Z}}) = 0$ for $i > 0$. So we get $H^i(S^1, \underline{\mathbb{Z}}) = \check{H}^i(\mathcal{U}, \underline{\mathbb{Z}})$, the cohomology of the Čech complex which has the form

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z} &\longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \longrightarrow \dots \\ (a, b) &\longmapsto (b - a, b - a) \end{aligned}$$

Thus we obtain:

$$H^i(S^1, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 1, \\ 0 & \text{if } i > 1. \end{cases}$$

17.2 Čech cohomology on paracompact Hausdorff spaces

'I'm basically mumbling right now.'

Theorem 17.2.1 · *If X is a paracompact Hausdorff space and $\mathscr{F}$ is an abelian sheaf on X, then there is an isomorphism*

$$\operatorname*{colim}_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathscr{F}) \xrightarrow{\cong} H^i(X, \mathscr{F}).$$

Definition 17.2.2 · Write $\mathbf{Cov}_X$ for the category of open covers of X, whose objects are the open covers $\mathcal{U} = \{U_i \hookrightarrow X\}_{i \in I}$ and whose maps are *refining maps*, that is, a map $\mathcal{V} = \{V_j \hookrightarrow X\}_{j \in J} \to \mathcal{U} = \{U_i \hookrightarrow X\}_{i \in I}$ is a function $\varphi : J \to I$ such that $V_j \subseteq U_{\varphi(j}$ for all $j \in J$.

Exercise 17.2.3 (Additional exercise 17.3) · The category $\mathbf{Cov}_X$ is not cofiltered, but the homotopy $(1, 0)$-category $h_0\mathbf{Cov}_X$ is. The objects of this category are the same as $\mathbf{Cov}_X$, but there is a unique arrow $\mathcal{V} \to \mathcal{U}$ if there exists a refining map $\mathcal{V} \to \mathcal{U}$, and no arrow otherwise.

Exercise 17.2.4 (Additional exercise 17.4) · Čech cohomology defines functors

$$\check{H}^i : h_0\mathbf{Cov}_X^{\mathrm{op}} \times \mathbf{PAb}(X) \to \mathbf{Ab}, \quad (\mathcal{U}, F) \mapsto \check{H}^i(\mathcal{U}, F).$$

Definition 17.2.5 · Define $\check{H}^i(X, F) := \operatorname{colim}_{\mathcal{U} \in h_0\mathbf{Cov}_X} \check{H}^i(\mathcal{U}, F)$.

Remark 17.2.6 · Note that Čech cohomology is contravariant in the open cover; the colimit of Definition 17.2.5 can be thought of as the Čech cohomology of F with respect to the 'finest' cover (such a cover might not exist).

write

LECTURE 18

Stratified spaces

> 'Now it's just three weeks of me having fun.'

18.1 Stratified spaces and exodormy

In Lecture 4, we discussed the following diagram:

$$\begin{array}{ccccc} \mathbf{Sh}(X) & \xrightarrow{\simeq} & \mathbf{LocalHomeo}_{/X} & & \\ \uparrow & & \uparrow & & \\ \mathbf{Sh}^{\mathrm{lc}}(X) & \xrightarrow[\simeq]{} & \mathbf{Cov}_{/X} & \xrightarrow[\simeq]{} & \mathbf{Fun}(\mathbf{B}\pi_1(X,x_0),\mathbf{Set}) \end{array}$$

(To avoid choosing a basepoint, we can replace $\mathbf{B}\pi_1(X, x_0)$ by the fundamental groupoid[1] $\Pi_1(X)$, whose objects are the points $x \in X$ and whose maps $x \to y$ are homotopy classes of paths from x to y.)

We will discuss in the following weeks in more detail the question: is there something in the top right spot? This will be partially answered by *exodromy*, but only for *constructible sheaves*. In this week's lecture, we discuss the topological setup, and next week we introduce constructible sheaves and discuss exodromy.

18.2 Stratifications

The most basic version of a stratification (a notion we will introduce momentarily) is a *filtration*.

Definition 18.2.1 · A *filtration* of a topological space X is a sequence of closed subspaces

$$\varnothing = Z_{-1} \subseteq Z_0 \subseteq Z_1 \subseteq \ldots \subseteq X$$

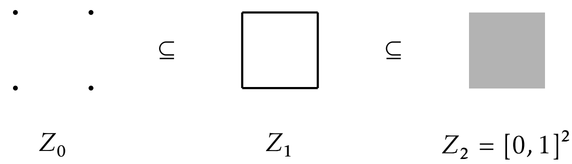
indexed by the natural numbers (and -1, but we always set $Z_{-1} := \varnothing$) such that $X = \bigcup_{i\in\mathbb{N}} Z_i$.

Example 18.2.2 · One may filter the n-cube $[0, 1]^n$ by

$$Z_i := \{ (x_1, \ldots, x_n) \mid \#\{x_j \neq 0, 1\} \leqslant i \}.$$

The picture for $n = 2$ is as follows:

[1] If you want to be fancy, this is the homotopy 1-category of the fundamental ∞-groupoid of X (modelled as a Kan complex by the singular simplicial set of X).



Example 18.2.3 · Generalising the previous example, a CW-complex X is filtered by its skeleta:

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X$$

The i-skeleton X_i is the union of the cells of dimension $\leqslant i$.

It is often convenient to describe the locally closed subsets $Z_i \setminus Z_{i-1}$ instead of the subsets Z_i. This is captured in the following definition.

Definition 18.2.4 · An $\mathbb{N}$-*stratification* of a topological space X is a decomposition

$$X = \coprod_{i\in\mathbb{N}} X_i$$

into locally closed subsets X_i such that $\overline{X_i} \subseteq X_{\leqslant i} := \bigcup_{j\leqslant i} X_j$. The disjoint union here is a disjoint union of *sets*; X need not have the disjoint union topology with respect to the X_i.

Lemma 18.2.5 · *Let $X = \coprod_{i\in\mathbb{N}} X_i$ be a decomposition of a topological space into subsets X_i. Then the following are equivalent:*

(i) the subsets X_i are locally closed and $\overline{X_i} \subseteq X_{\leqslant i}$ (the decomposition is an $\mathbb{N}$-stratification in the above sense);

(ii) the subsets $X_{\leqslant i}$ are closed.

Proof. If the latter holds, then the $X_i = X_{\leqslant i} \setminus X_{\leqslant i-1}$ are locally closed and since X_i is contained in the closed subset $X_{\leqslant i}$, so is its closure. Conversely, if the former statement holds, then

$$\overline{X_{\leqslant i}} = \bigcup_{j\leqslant i} \overline{X_j} \subseteq \bigcup_{j\leqslant i} X_{\leqslant j} = X_{\leqslant i},$$

so the $X_{\leqslant i}$ are closed. □

Lemma 18.2.6 · *Filtrations of a topological space X correspond bijectively to $\mathbb{N}$-stratifications of X via the maps*

$$(Z_0 \subseteq Z_1 \subseteq \ldots) \mapsto (X = \coprod_{i\in\mathbb{N}} Z_i \setminus Z_{i-1}),$$
$$(X_{\leqslant 0} \subseteq X_{\leqslant 1} \subseteq \ldots) \leftarrow\!\shortmid (X = \coprod_{i\in\mathbb{N}} X_i).$$

Proof. If $(Z_0 \subseteq Z_1 \subseteq \ldots)$ is a filtration of X, then $Z_i \setminus Z_{i-1}$ is locally closed, and $X_{\leqslant i} = \bigcup_{j\leqslant i} Z_j \setminus Z_{j-1} = Z_i$ is closed. Conversely, if $X = \coprod_{i\in\mathbb{N}} X_i$ is an $\mathbb{N}$-stratification, then the $X_{\leqslant i}$ are closed, and $\bigcup_{i\in\mathbb{N}} X_{\leqslant i} = X$. Clearly, $X_{\leqslant i} \setminus X_{\leqslant i-1} = X_i$. We have thus seen that the maps are well-defined and are inverses. □

It is often useful to allow other posets than $\mathbb{N}$.

Definition 18.2.7 · The *Alexandroff topology* on a poset P is the topology whose opens are the *upwards closed subsets* (also called *cosieves*): those subsets $U \subseteq P$ such that if $p \in U$ and $p \leqslant q \in P$, then $q \in U$.

Remark 18.2.8 · The Alexandroff topology is a topology; in fact, both arbitrary unions and arbitrary intersections of cosieves are cosieves. The closed subsets are the *downward closed subsets* (*sieves*), which also form a topology: the Alexandroff topology on P^{op}.

Example 18.2.9 · For $P = [1] = \{0 < 1\}$, the opens in the Alexandroff topology on P are $\varnothing$, $\{1\}$ and $\{0, 1\}$, so this is the Sierpiński space.

Example 18.2.10 · The sets $P_{\geqslant p} = \{ q \in P \mid q \geqslant p \}$ are open, and $P_{\leqslant p}$ is closed. Likewise, $P_{>p}$ is open and $P_{<p}$ is closed. The singletons $\{p\} = P_{\geqslant p} \cap P_{\leqslant p}$ are locally closed.

Definition 18.2.11 · Let X be a topological space and P a poset. Then a *P-stratification* on X is a continuous map $f : X \to P$ (where P is endowed with the Alexandroff topology). A *stratification* on X is a P-stratification for some P.

Example 18.2.12 · If $P = \mathbb{N}$, then a continuous map $f : X \to \mathbb{N}$ is a the same thing as an $\mathbb{N}$-stratification in the sense of Definition 18.2.4: the closed subsets (sieves) on $\mathbb{N}$ are exactly $\mathbb{N}_{\leqslant i}$ for some $i \in \mathbb{N}$, and $\varnothing$ and $\mathbb{N}$.

Example 18.2.13 · If $P = [1]$, we saw on Homework 1 that a P-stratification $f : X \to P$ is given by an open subsets $U := f^{-1}(1)$ (with closed complement $Z := f^{-1}(0)$)

Definition 18.2.14 · Given a map $f : X \to P$ (not necessarily continuous) from a topological space X to a poset P, we write

$$X_{\geqslant p} := f^{-1}(P_{\geqslant p}), \quad X_{>p} := f^{-1}(P_{>p}), \quad X_{\leqslant p} := f^{-1}(P_{\leqslant p}), \quad X_{<p} := f^{-1}(P_{<p}).$$

The set $X_p := f^{-1}(p)$ is called the *pth stratum*; it is locally closed if f is continuous.

Remark 18.2.15 · Continuity of $f : X \to P$ implies

$$f(\overline{X_p}) \subseteq \overline{f(X_p)} = \overline{\{p\}} = P_{\leqslant p},$$

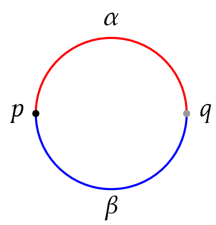
so $\overline{X_p} \subseteq X_{\leqslant p}$. The converse holds if $P = \mathbb{N}$ or if P is finite, but not in general (see Additional exercise 18.4).

example n-cube

Example 18.2.16 · The unit interval $I = [0, 1]$ can be stratified over $[1]$ as in the following picture:

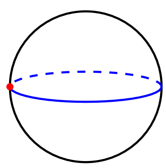
sending the left endpoint 0 (red) to 0 and the rest of the interval to 1.

Example 18.2.17 · The circle can be stratified as in the picture



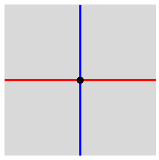
over the poset with elements $\{p, q, \alpha, \beta\}$ with $p < \alpha$, $p < \beta$, $q < \alpha$ and $q < \beta$.

Example 18.2.18 · We can stratify the 2-sphere over $[2] = \{0 < 1 < 2\}$ as in the following picture:



sending the red point to 0, the blue equator to 1, and the two hemispheres to 2. We can also stratify the 2-sphere over the poset with elements $\{0, 1, 2, 2'\}$ with $0 < 1$, $1 < 2$ and $1 < 2'$ by sending the upper hemisphere to 2 and the lower hemisphere to $2'$.

Example 18.2.19 · The plane $\mathbb{R}^2$ can be stratified as in the picture



over the poset with elements $\{0, 1, 1', 2\}$ with $0 < 1$, $0 < 1'$, $1 < 2$ and $1' < 2$ by sending the origin to 0, the x-axis minus the origin to 1, the y-axis minus the origin to $1'$ and the rest to 2.

Remark 18.2.20 · The examples above all satisfy extra properties:

(i) if $p \leqslant q$, then $X_p \cap \overline{X_q} \neq \varnothing$;

(ii) $\overline{X_p}$ is a union of strata ($X_p \cap \overline{X_q} \neq \varnothing$ implies $X_p \subseteq \overline{X_q}$).

Together, these properties are equivalent to $\overline{X_q} = X_{\leqslant q}$; this condition is called the 'axiom of the frontier'. We will not explicitly impose this condition, but many authors *do* assume the axiom of the frontier or other additional aximos.

write

18.3 Conical stratifications

Definition 18.3.1 · The *left cone* $\mathscr{C}^{\triangleleft}$ on a category $\mathscr{C}$ is the category with objects $\mathscr{C} \amalg \{-\infty\}$ and maps

$$\mathrm{Hom}_{\mathscr{C}^{\triangleleft}}(x, y) = \begin{cases} \mathrm{Hom}_{\mathscr{C}}(x, y) & \text{if } x, y \in \mathscr{C}, \\ * & \text{if } x = -\infty, \\ \varnothing & \text{if } x \in \mathscr{C}, y = -\infty. \end{cases}$$

Dually, the *right cone* $\mathscr{C}^{\triangleright}$ of $\mathscr{C}$ is the category $((\mathscr{C}^{\mathrm{op}})^{\triangleleft})^{\mathrm{op}}$.

The left cone of a category $\mathscr{C}$ is obtained by formally adjoining an initial object to $\mathscr{C}$; dually, the right cone of $\mathscr{C}$ is $\mathscr{C}$ where a terminal object is formally adjoined.

Remark 18.3.2 · If $\mathscr{C}$ is a preorder, that is, if $\#\,\mathrm{Hom}_{\mathscr{C}}(x, y) \leqslant 1$ for all $x, y \in \mathscr{C}$, then so is the left cone $\mathscr{C}^{\triangleleft}$. If $\mathscr{C}$ is a poset (that is, a preorder such that $\mathrm{Hom}_{\mathscr{C}}(x, y) \times \mathrm{Hom}_{\mathscr{C}}(y, x) \neq \varnothing$ implies $x = y$), then so is the left cone $\mathscr{C}^{\triangleleft}$.

We will only deal with left cones of posets in this and next week's lecture, but since the definition makes sense for all categories, we have presented it in this generality.

Example 18.3.3 · For the totally ordered set $[n] = \{0 < 1 < \ldots < n\}$ with $n + 1$ elements, we have

$$[n]^{\triangleleft} \cong [n + 1] \cong [n]^{\triangleright}.$$

Definition 18.3.4 · The *cone* CX of a topological space X is the space $CX := (X \times \mathbb{R}_{>0}) \cup \{*\}$ where $U \subseteq CX$ is open if and only if $U \cap (X \times \mathbb{R}_{>0})$ is open and if $* \in U$ implies that $X \times (0, \varepsilon) \subseteq U$ for some $\varepsilon > 0$.

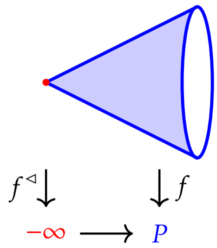
Remark 18.3.5 · Usually, one defines the cones as the pushout $X \times \mathbb{R}_{\geqslant 0} \amalg_{X \times \{0\}} \{*\}$. Then the second condition becomes: if $* \in U$, then for all $x \in X$ there exists an open neighbourhood $V_x \subseteq X$ of x and an $\varepsilon_x > 0$ such that $V_x \times (0, \varepsilon_x) \subseteq U$.

The natural map $(X \times \mathbb{R}_{\geqslant 0}) \amalg_{X \times \{0\}} \{*\} \to CX$ is a homeomorphism if X is compact, but not when $X = \mathbb{R}$ for instance.

Definition 18.3.6 · If $f : X \to P$ is a stratification, define the *left cone* $f^{\triangleleft} : CX \to P^{\triangleleft}$ of f by $* \mapsto -\infty$ and $(x, t) \mapsto f(x)$ for $(x, t) \in X \times \mathbb{R}_{>0}$.

This definition makes sense as the point $*$ is closed, and contained in the closure of $Z \times \mathbb{R}_{>0}$ for any closed subset $Z \subseteq X$.

The picture for the left cone of a P-stratification of the circle is as follows:



Definition 18.3.7 · A stratification $f : X \to P$ is *conical* if every point $x \in X$ has an open neighbourhood $U \subseteq X_{\leqslant p}$ where $p := f(x)$ such that $f|_U : U \to P_{\geqslant p}$ is isomorphic to $Z \times CY$ for some $P_{>p}$-stratified space Y and a path connected and locally path connected [2] space Z. Such a neighbourhood U is called a *basic neighbourhood*.

Note for this definition that $(P_{>p})^{\triangleleft} \cong P_{\geqslant p}$.

We state the following fact without explaining what we mean precisely.

Proposition 18.3.8 · *Stratifications by closed embedded submanifolds are conical.*

Examples and counterexamples

write

[2]This should probably include 'semi-locally simply connected'.

LECTURE 19

Constructible sheaves, exit paths, exodromy

'So we run into the problem that category theory is hard.'

19.1 Constructible sheaves

write

19.2 Exit path category

write

19.3 Exodromy

write

LECTURE 20

Outlook

20.1 Comparison between cohomology theories

> 'I should say something at some point,
> let me not do that yet.'

In this section, we compare different definitions of cohomology theories for topological spaces. Specifically, we will compare sheaf cohomology, Čech cohomology, de Rham cohomology, singular cohomology and the 'Eilenberg–MacLane approach' to cohomology (also called the 'homotopy construction of cohomology'). The conclusion will be the picture in Figure 20.1. The arrows' labels point to the comparison results.

Proposition 20.1.1 · *If X is paracompact Hausdorff, then $\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$ for all abelian sheaves $\mathcal{F}$ on X. Moreover, if $\mathcal{U}$ is an open cover of X such that each intersection $U_{i_0} \cap \cdots \cap U_{i_k}$ is a disjoint union of contractible spaces, then $\check{H}^i(\mathcal{U}, \underline{A}) \cong H^i(X, \underline{A})$ for every abelian group A.*

These results were proven in Theorem 17.2.1, Corollary 17.1.7 and Example 17.1.8.

Proposition 20.1.2 · *If X is a C^∞-manifold (assumed to be paracompact), then $H^i_{\mathrm{dR}}(X) \cong H^i(X, \underline{\mathbb{R}})$.*

Proof. The de Rham complex

$$0 \to \underline{\mathbb{R}} \to \Omega^0_X \to \Omega^1_X \to \ldots$$

(see Example 14.1.4) is an exact sequence of sheaves: locally, every closed i-from is exact (Poincaré lemma). Since Ω^i_X is a sheaf of $C^\infty(-, \mathbb{R})$-modules, it is soft, so the above is an acyclic resolution of $\underline{\mathbb{R}}$. □

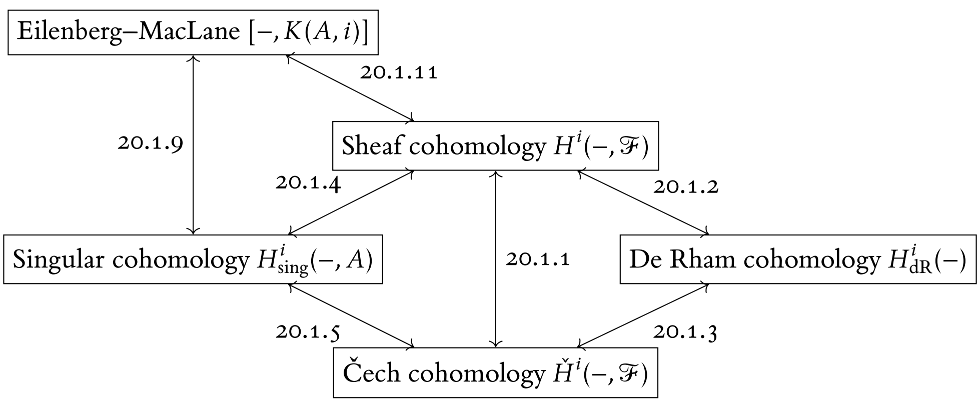


Figure 20.1 · Comparison of cohomology theories

Proposition 20.1.3 ([BT82, Theorem 8.9]) · *If X is a manifold, then $H^i_{\mathrm{dR}}(X) \cong \check{H}^i(\mathcal{U}, \underline{\mathbb{R}})$ for any cover $\mathcal{U}$ such that each intersection $U_{i_0} \cap \cdots \cap U_{i_k}$ is a disjoint union of contractible spaces.*

Proposition 20.1.4 ([Voi02, Theorem 4.47]) · *If X is locally contractible, then $H^i_{\mathrm{sing}}(X, A) \cong H^i(X, \underline{A})$ for every abelian group A.*

Proof. Let $\mathscr{C}^i_{\mathrm{sing}}(A)$ be the sheafification of the presheaf $U \mapsto C^i_{\mathrm{sing}}(U, A)$ sending an open to the singular i-cochains[1] on U with coefficients in A. The sequence of sheaves

$$0 \to \underline{A} \to \mathscr{C}^0_{\mathrm{sing}}(A) \to \mathscr{C}^1_{\mathrm{sing}}(A) \to \dots$$

is exact as X is locally contractible. Each presheaf C^i_{sing} is flasque: extend by taking any simplex $\Delta^i \to X$ that does not land in U to 0. Now $\mathscr{C}^i_{\mathrm{sing}}(A)(U)$ is C^i_{sing} modulo locally trivial cochains, so $\mathscr{C}^i_{\mathrm{sing}}(A)$ is flasque as well. Finally, the map

$$C^\bullet_{\mathrm{sing}}(X, A) \to \Gamma(X, \mathscr{C}^\bullet_{\mathrm{sing}}(A)) = C^\bullet_{\mathrm{sing}}/\{\text{locally trivial cochains}\}$$

is a quasi-isomorphism (this result is known as the 'theorem on small chains'). □

Proposition 20.1.5 · *If X admits a cover $\mathcal{U}$ such that each intersection $U_{i_0} \cap \cdots \cap U_{i_k}$ is a disjoint union of contractible spaces, then $H^i_{\mathrm{sing}}(X, A) \cong \check{H}^i(X, \underline{A})$ for every abelian group A.*

To state the comparison between singular cohomology and the 'Eilenberg–MacLane approach' (what [Hat02] calls the 'homotopy construction of cohomology'), we have to introduce some notions.

Definition 20.1.6 · Let (G, n) be a pair of a natural number n and a group G (which should be abelian if $n \geqslant 2$). Then an *Eilenberg–MacLane space $K(G, n)$* is a pointed space such that

$$\pi_i(K(G, n)) \cong \begin{cases} G & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 20.1.7 · The circle S^1 is a $K(\mathbb{Z}, 1)$ (as can be seen from the long exact sequence of the fibration $\mathbb{Z} \to \mathbb{R} \to S^1$).

Theorem 20.1.8 ([Hat02, Proposition 4.30]) · *For every pair (G, n) as in Definition 20.1.6, there exists a $K(G, n)$, and any two $K(G, n)$'s are weakly homotopy equivalent.*

Proposition 20.1.9 (representability of singular cohomology [Hat02, Theorem 4.57]) · *If X is a CW-complex, then $[X, K(A, i)] \cong H^i_{\mathrm{sing}}(X, A)$ for every abelian group A.*

Remark 20.1.10 · Since singular homology $H^i(-, A) : \mathbf{Top} \to \mathbf{Ab}$ is homotopy invariant, it can be seen as a functor $H^i(-, A) : \mathbf{Ho}(\mathbf{Top}) \to \mathbf{Ab}$ on the homotopy category of topological spaces. Proposition 20.1.9 says that this latter functor is represented by the Eilenberg–MacLane space $K(A, i)$.

Proposition 20.1.11 · *If X is paracompact Hausdorff, then $[X, K(A, i)] \cong H^i(X, \underline{A})$ for every abelian group A.*

We saw this for $K(\mathbb{Z}, 1) \simeq S^1$ in Homework 8, Exercise 2. We are not sure about a reference for the general statement.

We have seen that the different cohomology theories agree on sufficiently nice spaces. In general, however, the answers can be genuinely different for pathological spaces, as illustrated by the following example.

[1] This can be taken to be the set of functions from the set $\mathrm{Hom}_{\mathbf{Top}}(\Delta^i, U)$ to A, which inherits an abelian group structure from A. Here Δ^i denotes the standard topological i-simplex.

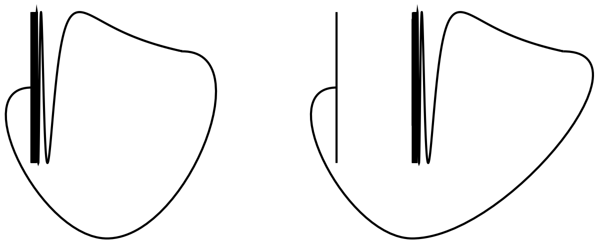


Figure 20.2 · The Warsaw circle and the separated Warsaw circle

Example 20.1.12 · Let X denote the Warsaw circle (the union of the topologist's sine curve $(x, 1/\sin x)$ for all $x \in (0,1]$, a straight line segment from $(0,1)$ to $(0,-1)$ and an arc connecting the right endpoint of the sine curve to the line segment at the y-axis) and Y the 'separated' Warsaw circle (where the line segment at the y-axis is moved to the left) as displayed in Figure 20.2.

Then we have $H^i_{\mathrm{sing}}(X, A) = 0$ for all $i > 0$: any i-simplex $\Delta^i \to X$ factors through Y since the standard topological simplex is path connected, so $H^i_{\mathrm{sing}}(X, A) \cong H^i_{\mathrm{sing}}(Y, A)$ for $i > 0$, and Y is contractible so its higher singular cohomology vanishes.

On the other hand, there is a natural map $X \to S^1$ which projects the sine curve to the x-axis, and this maps is not nullhomotopic since it cannot be lifted to $\mathbb{R}$. By the isomorphism $H^1(X, \underline{\mathbb{Z}}) \cong [X, S^1]$ of Homework 8, Exercise 2, we see that $H^1(X, \underline{\mathbb{Z}})$ does not vanish.

We conclude that sheaf cohomology gives the 'best answer':

- The Eilenberg–MacLane approach works for spaces with many maps out of it (paracompact Hausdorff).
- Singular cohomology works for spaces with many maps into it (CW-complexes).
- De Rham cohomology works for manifolds (what we might call locally trivial spaces?).

The introduction of Lurie's preprint *On Infinity Topoi* [Lur03] also discusses these various notions of cohomology; we recommend taking a look. The next section discusses the novel ideas from Lurie's work.

20.2 Stacks and higher topoi

The goal of Lurie's preprint *On Infinity Topoi* [Lur03] was to generalise the set $[X, K(G, n)]$ of homotopy classes of maps into an Eilenberg–MacLane space to the set $[X, Y]$ of homotopy classes of maps into an arbitrary space/homotopy type/Kan complex/∞-groupoid/anima[2] Y, using an 'internal' definition in terms of sheaves on X.

Definition 20.2.1 · A *sheaf of spaces* (also called an ∞*-stack*) on a topological space X is a functor

$$\mathcal{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Spc}$$

into the ∞-category of spaces (often also denoted $\mathcal{S}$) such that for every open cover $U = \bigcup_{i \in I} U_i$, the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \rightrightarrows \prod_{i_0, i_1 \in I} \mathcal{F}(U_{i_0} \cap U_{i_1}) \overset{\textstyle\rightarrow}{\rightrightarrows} \prod_{i_0, i_1, i_2 \in I} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap U_{i_2}) \overset{\textstyle\rightrightarrows}{\rightrightarrows} \cdots$$

realises $\mathcal{F}(U)$ as the homotopy limit (the limit in the ∞-categorical sense [Lur09, Definition 1.2.13.4]) of the rest of the diagram.

[2] These words all mean the same thing: they are presentations of topological spaces up to weak homotopy equivalence. When we say 'space' in this section, this is what we mean, and we always use the adjective 'topological' when talking about topological spaces. If Y is a Kan complex, then $[X, Y]$ should be interpreted as the set $[X, |Y|]$ of homotopy classes of maps into the geometric realisaton of Y.

If we work with the ∞-category $\mathbf{Spc}_{\leqslant n}$ of n-truncated spaces (that is, spaces with vanishing homotopy groups above degree n), we only need to consider the diagram up to the $(n+1)$-fold product. For instance, for $n = 0$ we have $\mathbf{Spc}_{\leqslant 0} \simeq \mathbf{Set}$, and we only need to look at the part

$$\mathscr{F}(U) \longrightarrow \prod_{i_0 \in I} \mathscr{F}(U_{i_0}) \rightrightarrows \prod_{i_0, i_1 \in I} \mathscr{F}(U_{i_0} \cap U_{i_1})$$

as we indeed did before for sheaves of sets. (This is related to the fact that $(-)^\sharp = \check{H}^0(X, \check{H}^0(X, -))$: after one iteration of $\check{H}^0(X, -)$, the map from $\mathscr{F}(U)$ to the equaliser of $\prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$ is (-1)-truncated (that is, injective). After the second iteration, it is (-2)-truncated, so an isomorphism.)

The ∞-category of n-truncated spaces can alternatively be presented as the ∞-category of n-groupoids (for $n \geqslant -2$). In the low degrees, the ∞-category of (-2)-groupoids is equivalent to the terminal category; the ∞-category of (-1)-groupoids is equivalent to the walking arrow category $\varnothing \to *$; and the ∞-category is equivalent o the category of sets.

Example 20.2.2 · A sheaf of 1-truncated spaces is a *stack* (in 1-groupoids): it is a functor $\mathrm{Open}(X)^{\mathrm{op}} \to$ **Gpd** (as ∞-categories) with a gluing condition.

Example 20.2.3 · The association

$$U \mapsto \{\text{rank } n \text{ locally constant sheaves on } U\}^{\cong}$$

sending an open U to the groupoid core of the full subcategory of rank n locally constant sheaves is a stack. It is the constant sheaf $\mathbf{B}\,\mathrm{GL}_n(\underline{\mathbb{Z}})$. The sheaf condition says: if we have a family $(\mathscr{G}_i)_{i \in I}$ of locally constant sheaves and isomorphisms $(\varphi_{ij} : \mathscr{G}_j|_{U_i \cap U_j} \cong \mathscr{G}_i|_{U_i \cap U_j})_{i,j \in I}$ such that $\varphi_{ij}|_{U_i \cap U_j \cap U_k} \circ \varphi_{jk}|_{U_i \cap U_j \cap U_k} = \varphi_{ik}|_{U_i \cap U_j \cap U_k}$, then we can glue the $\mathscr{G}_i$ to a sheaf $\mathscr{G}$ on U – this is just 'gluing sheaves'!

Remark 20.2.4 · The classical literature about (1-)stacks translates everything in concrete statements about 1-categories.

Theorem 20.2.5 ([Lur09, Theorem 7.1.0.1]) · *If X is a paracompact topological space and K is a space, then the functor*

$$\mathrm{R}\Gamma(X, -) : \mathbf{Sh}(X, \mathbf{Spc}) \to \mathbf{Sh}(*, \mathbf{Spc}) \simeq \mathbf{Spc}$$

induced by $X \to *$ satisfies

$$\pi_0 \mathrm{R}\Gamma(X, \underline{K}) \cong [X, K].$$

The proof of this theorem is finished on page 705 of [Lur09].

20.3 Relative Poincaré duality

write

'Should we go back to earth?'

APPENDIX A

Computing sheaf cohomology and higher pushforwards

In this appendix, we summarise some strategies for computing sheaf cohomology and higher pushforwards.

A.1 Sheaf cohomology

Compact spaces

Lemma A.1.1 (Lemma 13.2.4) · *Let $\mathcal{F}$ be an abelian sheaf on a compact space X. Then $H^i(X,\mathcal{F}) \cong H^i_c(X,\mathcal{F})$ for all $i \geqslant 0$.*

This result has the following consequences for computing the sheaf cohomology of a space X:

- If X is compact, we can use both strategies for ordinary cohomology and strategies for compactly supported cohomology to compute either one.
- If it is possible to relate X to a compact space K (one example we encountered was the embedding $\mathbb{R}^n \cong (0,1)^n \hookrightarrow [0,1]^n$), it might be possible to use strategies for compactly supported cohomology of K to say something about ordinary cohomology of X.

Open and closed subsets

Lemma A.1.2 · *Let $i : Z \hookrightarrow X$ be a closed subset of a topological space X and let $\mathcal{F}$ be a sheaf on Z. Then we have*

$$H^i(X, i_*\mathcal{F}) \cong H^i(Z,\mathcal{F})$$

for all $i \geqslant 0$.

Lemma A.1.3 (Additional exercise 13.1(d)) · *Let $j : U \hookrightarrow X$ be an open subset of a locally compact Hausdorff space X and let $\mathcal{F}$ be a sheaf on U. Then we have*

$$H^i_c(X, j_!\mathcal{F}) \cong H^i_c(U,\mathcal{F})$$

for all $i \geqslant 0$.

Lemma A.1.4 (Homework 3, Exercise 4(d)) · *Let $\mathcal{F}$ be an abelian sheaf on a locally compact Hausdorff space X and let $j : U \hookrightarrow X$ be an open subset with closed complement $i : Z \hookrightarrow X$. Then there is a short exact sequence of sheaves of the form*

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0.$$

Note that $j^*\mathcal{F} = \mathcal{F}|_U$ and $i^*\mathcal{F} = \mathcal{F}|_Z$.

Corollary A.1.5 (open–closed sequence, Lemma 14.2.5) · *Let $\mathcal{F}$ be an abelian sheaf on a locally compact Hausdorff space X and let $U \subseteq X$ be an open subset with closed complement Z. Then there is a long exact sequence of the form*

$$\ldots \to H^i_c(U, \mathcal{F}|_U) \to H^i_c(X, \mathcal{F}) \to H^i_c(Z, \mathcal{F}|_Z) \to H^1(U, \mathcal{F}|_U) \to \ldots .$$

Mayer–Vietoris

The Mayer–Vietoris sequence for ordinary sheaf cohomology:

Proposition A.1.6 (Mayer–Vietoris sequence, Homework 7, Exercise 1(c)) · *Let $\mathcal{F}$ be an abelian sheaf on a topological space X and let U and V be open subsets of X. Then there is a long exact sequence of the form*

$$\ldots \to H^i(U \cup V, \mathcal{F}) \to H^i(U, \mathcal{F}) \oplus H^i(V, \mathcal{F}) \to H^i(U \cap V, \mathcal{F}) \to H^{i+1}(U \cup V, \mathcal{F}) \to \ldots .$$

There is a variant for compactly supported sheaf cohomology. Note that the order is flipped:

Proposition A.1.7 (compactly supported Mayer–Vietoris sequence, Additional exercise 13.2) · *Let $\mathcal{F}$ be an abelian sheaf on a locally compact Hausdorff space X and let U and V be open subsets of X. Then there is a long exact sequence of the form*

$$\ldots \to H^i_c(U \cap V, \mathcal{F}) \to H^i_c(U, \mathcal{F}) \oplus H^i_c(V, \mathcal{F}) \to H^i_c(U \cup V, \mathcal{F}) \to H^{i+1}_c(U \cap V, \mathcal{F}) \to \ldots .$$

Coefficients

Lemma A.1.8 (Corollary 16.1.2) · *If X is a locally compact Hausdorff space with compactly supported cohomology with $\underline{\mathbb{Z}}$ coefficients given by*

$$H^i_c(X, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

then we have

$$H^i_c(X, \underline{A}) = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

for any abelian group A.

Homotopy invariance

Theorem A.1.9 (Vietoris–Begle mapping theorem, Theorem 16.1.4) · *Let $f : Y \to X$ be a proper map such that*

$$H^i(f^{-1}(x), \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases}$$

for all $x \in X$. Then for all abelian sheaves $\mathcal{F}$ on X, the maps

$$H^i(X, \mathcal{F}) \to H^i(Y, f^*\mathcal{F})$$

are isomorphisms for all $i \in \mathbb{N}$.

Corollary A.1.10 (Corollary 16.1.7) · *If $f, g : Y \to X$ are homotopic maps, then the maps*

$$f^*, g^* : H^i(X, \underline{A}) \to H^i(Y, \underline{A})$$

agree for all $i \geqslant 0$ and all abelian groups A.

Corollary A.1.11 (Corollary 16.1.8) · *If $f : X \to Y$ is a homotopy equivalence (of paracompact Hausdorff or locally compact Hausdorff spaces), then*

$$f^* : H^i(X, \underline{A}) \xrightarrow{\cong} H^i(Y, \underline{A})$$

is an isomorphism for any abelian group A. In particular, if X and Y are homotopy equivalent, then $H^i(X, \underline{A}) \cong H^i(Y, \underline{A})$ for any A.

In particular, if X is contractible (and paracompact Hausdorff or locally compact Hausdorff), then

$$H^i(X, \underline{A}) = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{if } i > 0 \end{cases}$$

for any abelian group A.

Čech cohomology

Proposition A.1.12 (Corollary 17.1.7) · *Let X be a topological space with an open cover $\mathcal{U} = \{U_i \hookrightarrow X\}_{i \in I}$ and let $\mathcal{F}$ be an abelian sheaf on X. If $H^i(U_{i_0,\dots,i_n}, \mathcal{F}) = 0$ for all $i > 0$, all $n \geqslant 0$ and all $i_0, \dots, i_n \in I$, then the map*

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$$

is an isomorphism.

Corollary A.1.13 (Example 17.1.8) · *If all intersections $U_{i_0,\dots,i_n}$ are disjoint unions of contractibles, then $H^i(U_{i_0,\dots,i_n}, \underline{\mathbb{Z}}) = 0$ for all $i > 0$, so $\check{H}^i(\mathcal{U}, \underline{\mathbb{Z}}) = H^i(X, \underline{\mathbb{Z}})$.*

Proposition A.1.14 (Theorem 17.2.1) · *Let $\mathcal{F}$ be an abelian sheaf on a paracompact Hausdorff space X. Then there is an isomorphism*

$$\operatorname*{colim}_{\mathcal{U} \in h_0 \mathbf{Cov}_X} \check{H}^i(\mathcal{U}, \mathcal{F}) \xrightarrow{\cong} H^i(X, \mathcal{F}).$$

A.2 Higher pushforwards

The following two lemmas express that the (proper) pushforward is the relative version of (compactly supported) cohomology.

Lemma A.2.1 (Lemma 14.3.4) · *Let $\mathcal{F}$ be an abelian sheaf on X and let $f : X \to *$ denote the unique map. Then $H^i(X, \mathcal{F}) = \mathrm{R}^i f_* \mathcal{F}$.*

Lemma A.2.2 (Exercise 15.2.6) · *Let $\mathcal{F}$ be an abelian sheaf on a Hausdorff space X and let $f : X \to *$ denote the unique map. Then $H^i_c(X, \mathcal{F}) = \mathrm{R}^i f_! \mathcal{F}$.*

Theorem A.2.3 (Theorem 15.3.1) · *Let $f : Y \to X$ be a proper map and assume that X and Y are either both paracompact Hausdorff or both locally compact Hausdorff. For an abelian sheaf $\mathcal{F}$ on Y, the natural map*

$$(\mathrm{R}^i f_* \mathcal{F})_x \to H^i(f^{-1}(x), \mathcal{F})$$

is an isomorphism for all $x \in X$ and for all $i \geqslant 0$.

Theorem A.2.4 (Theorem 15.3.3) · *Let $: : Y \to X$ be a map of locally compact Hausdorff spaces. For an abelian sheaf $\mathcal{F}$ on Y, the natural map*

$$(\mathrm{R}^i f_! \mathcal{F})_x \to H^i_c(f^{-1}(x), \mathcal{F})$$

is an isomorphism for all $x \in X$ and all $i \geqslant 0$.

Corollary A.2.5 (proper base change theorem, Corollary 15.3.6) · *Let the diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{\hat{f}} & X' \\ {\scriptstyle \hat{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

be a pullback square in **Top**. *Then:*

(i) If f is a proper map and X and X' are either both paracompact Hausdorff or both locally compact Hausdorff, then there are canonical isomorphisms

$$g^* \circ \mathrm{R}^i f_* \xrightarrow{\cong} \mathrm{R}^i \hat{f}_* \circ \hat{g}^* : \mathbf{Ab}(Y) \to \mathbf{Ab}(X).$$

(ii) If X, X' and Y' are locally compact Hausdorff, then there are canonical isomorphisms

$$g^* \circ \mathrm{R}^i f_! \xrightarrow{\cong} \mathrm{R}^i \hat{f}_! \circ \hat{g}^* : \mathbf{Ab}(Y) \to \mathbf{Ab}(X').$$

A.3 Cohomology of some spaces

The following results were computed in earlier lectures or in the homework exercises.

Corollary A.3.1 (Corollary 14.2.4) ·

(i) If X is either $\mathbb{R}$, $[0,1]$ or $[0,1)$ then the cohomology of X with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ is

$$H^i(X, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

(ii) The compactly supported cohomology of $\mathbb{R}$, $[0,1]$ and $[0,1)$ with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ is:

$$H^i_c(\mathbb{R}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 1, \\ 0 & \text{else}, \end{cases}$$
$$H^i_c([0,1], \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{else}, \end{cases}$$
$$H^i_c([0,1), \underline{\mathbb{Z}}) = 0 \qquad \text{for all } i.$$

Proposition A.3.2 (Cohomology of the sphere) · *The (compactly supported) cohomology of the n-sphere S^n with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ is*

$$H^i(S^n, \underline{\mathbb{Z}}) = H^i_c(S^n, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, n \\ 0, & \text{else}. \end{cases}$$

Note that the cohomology and compactly supported cohomology of any sheaf on S^n agree, by Lemma 13.2.4.

Proposition A.3.3 (Cohomology of Euclidean space) ·

(i) The cohomology of $\mathbb{R}^n$ with coefficients in the constant sheaf $\mathbb{Z}$ is

$$H^i(\mathbb{R}^n, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{else}, \end{cases}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$.

(ii) The compactly supported cohomology of $\mathbb{R}^n$ *with coefficients in the constant sheaf* $\mathbb{Z}$ *is*

$$H^i_c(\mathbb{R}^n, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}, & \text{if } i = n, \\ 0, & \text{else}, \end{cases}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$.

The following proposition genealises the cohomology of $\mathbb{R}^n$. It shows that the (compactly supported) cohomology with coefficients in the constant sheaf $\mathbb{Z}$ tracks the number of holes in $\mathbb{R}^n \smallsetminus \{x_1, \dots, x_m\}$. We treat the case $n = 1$ separately: the case $n > 1$ has nonvanishing cohomology in two distinct degrees, but these are the same degree for $i = 0$.

Proposition A.3.4 (Cohomology of Euclidean space with missing points) · *Let* $m, n \in \mathbb{Z}_{\geqslant 1}$, *and let* $x_1, \dots x_m$ *be distinct points in* $\mathbb{R}^n$.

(i) The cohomology of $\mathbb{R} \smallsetminus \{x_1, \dots, x_m\}$ *with coefficients in the constant sheaf* $\mathbb{Z}$ *is*

$$H^i(\mathbb{R} \smallsetminus \{x_1, \dots, x_m\}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}^{m+1}, & \text{if } i = 0, \\ 0, & \text{else}. \end{cases}$$

(ii) The cohomology of $\mathbb{R}^n \smallsetminus \{x_1, \dots, x_m\}$ *with coefficients in the constant sheaf* $\mathbb{Z}$ *is*

$$H^i(\mathbb{R}^n \smallsetminus \{x_1, \dots, x_m\}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}^m, & \text{if } i = n-1, \\ \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{else}, \end{cases}$$

for all $n \in \mathbb{Z}_{>1}$.

(iii) The compactly supported cohomology of $\mathbb{R} \smallsetminus \{x_1, \dots, x_m\}$ *with coefficients in the constant sheaf* $\mathbb{Z}$ *is*

$$H^i_c(\mathbb{R} \smallsetminus \{x_1, \dots, x_m\}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}^{m+1}, & \text{if } i = 1, \\ 0, & \text{else}. \end{cases}$$

(iv) The compactly supported cohomology of $\mathbb{R}^n \smallsetminus \{x_1, \dots, x_m\}$ *with coefficients in the constant sheaf* $\mathbb{Z}$ *is*

$$H^i_c(\mathbb{R}^n \smallsetminus \{x_1, \dots, x_m\}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}^m, & \text{if } i = 1 \\ \mathbb{Z}, & \text{if } i = n \\ 0, & \text{else}, \end{cases}$$

for all $n \in \mathbb{Z}_{>1}$.

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